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Stellar Electron Capture with Finite Temperature Relativistic Random Phase Approximation

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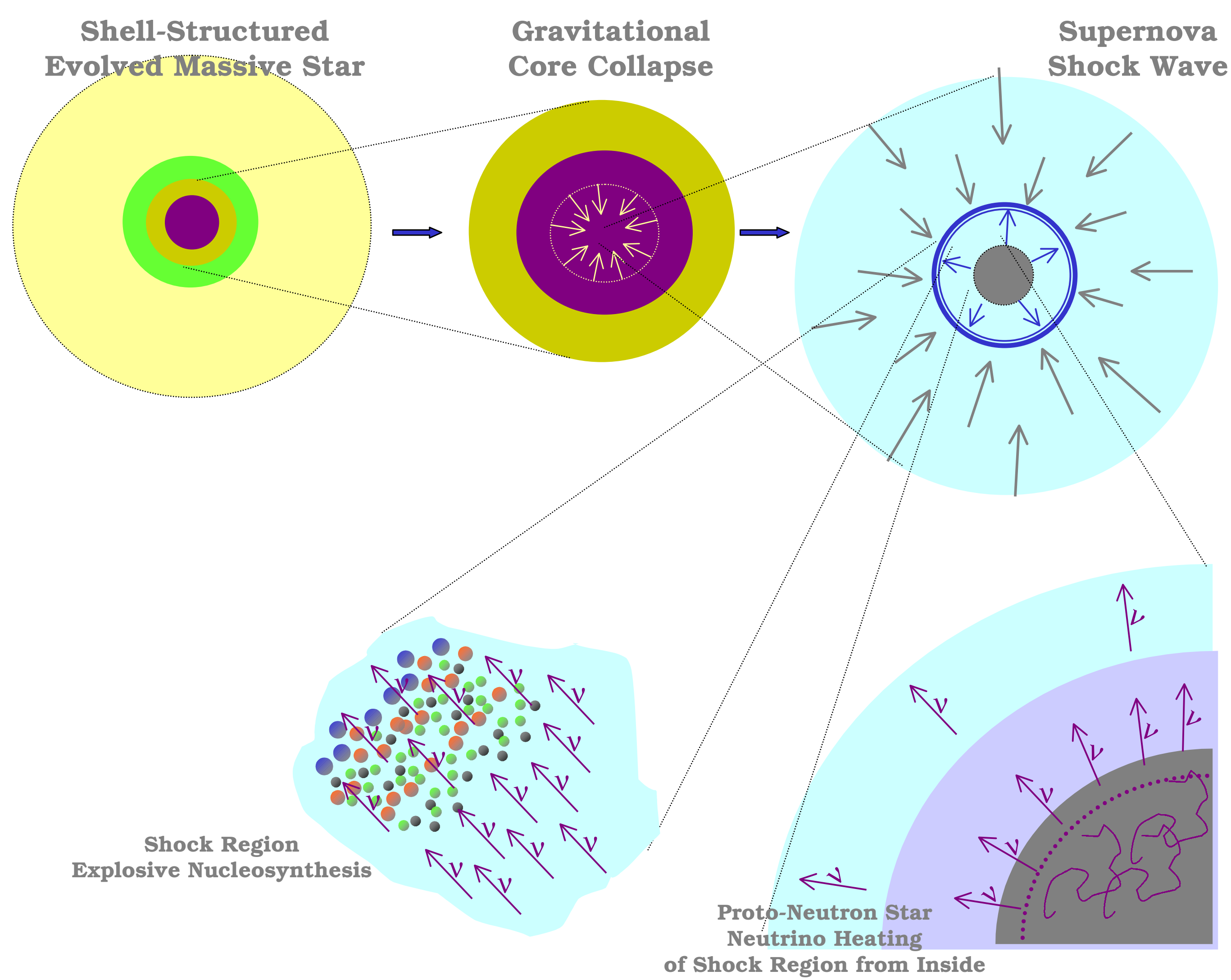
OUTLINE

Introduction — Electron capture in stellar evolution

Collapse of a massive star and a supernova explosion
— *from Langanke 2008, Acta Physica Polonica B,39.*



Introduction — Electron capture in stellar evolution

Collapse of a massive star and a supernova explosion
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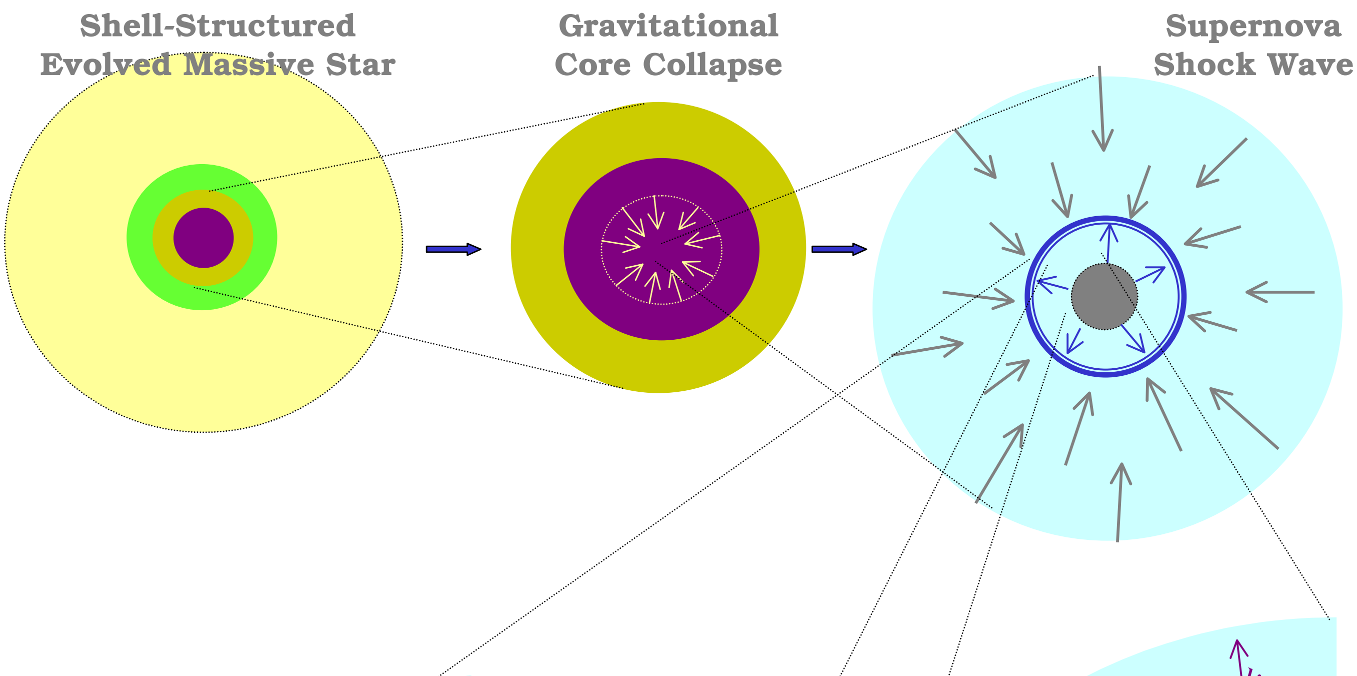


- The dynamics of core-collapse supernova $\Leftarrow \begin{cases} \text{core entropy} \\ \text{lepton to baryon ratio } Y_e \end{cases}$

$\Leftarrow$ weak interaction processes $\begin{cases} \beta \text{ decay} \\ \text{electron capture dominate !} \end{cases}$

Introduction — Electron capture in stellar evolution

Core Collapse type II supernova

◇ initial stage: $\rho \sim 10^{10}$ g/cm^3, $T = 300 - 800$ keV, μ_e of the same order as Q value

- Electrons are captured by iron range nuclei $A < 60$.
- Electron capture rates are sensitive to detailed GT distribution.

◇ higher densities and temperature: $\mu_e \gg Q$ value

- Electrons are captured by neutron rich nuclei with $A > 65$.
- Electron capture rates are mainly determined by the total GT strength and centroid energy.

◇ $\rho > 10^{11}$ g/cm^3, $\mu_e \sim 30$ MeV:

Forbidden transitions couldn't be neglected.

Introduction — Electron capture process

The electron capture on a nucleus ${}^{A}_{Z}X_{N}$

$$e^{-} + {}^{A}_{Z}X_{N} \rightarrow {}^{A}_{Z-1}X^{*}_{N+1} + \nu_e \tag{1}$$

- Cross section for a transition followed from Fermi's golden rule reads:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2} V^2 E_\nu^2 \frac{1}{2} \sum_{\text{lepton spins}} \frac{1}{2J_i + 1} \sum_{M_i M_f} |\langle f | \hat{H}_W | i \rangle|^2. \tag{2}$$

  V: the quantization volume; E_ν: energy of outgoing electron neutrino; H_W: the Hamiltonian of the weak interaction.

- The evaluation of the matrix elements $\langle f | \hat{H}_W | i \rangle$ is crucial for the calculation of electron capture cross sections. The initial and final states of nuclei are obtained from nuclear structure models.

Introduction — Theoretical description

Different nuclear models could be employed to extract information on the initial and final states for the investigation of the stellar electron capture.

- Independent Particle Model (IPM): for $A = 21 - 60$
  - ✓ the first standard tabulation of nuclear weak-interaction rates

    G. M. Fuller, W. A. Fowler, M. J. Newman, Ap. J. S. 42, 447 1980; 48, 279, 1982;
    G. M. Fuller, W. A. Fowler, M. J. Newman, Ap. J. 252, 715, 1982; 293, 1, 1985.
- Shell Model Monte Carlo(SMMC): for $A = 45 - 65$
  - ✓ for the first time determines in a microscopic way the Gamow-Teller contributions to the presupernova electron capture rates
  - ✓ take into account thermal effects

    D. J. Dean, K. Langanke, L. Chatterjee, P. B. Radha, and M. R. Strayer, Phys. Rev. C 58, 536, 1998.
- Large Scale Shell Model diagonalization (LSSM): for $A = 45 - 65$
  - ✓ an updated tabulation of weak interaction rates
  - ✓ reproduce the experimental GT^+ distributions

    K. Langanke, G. Martinez-Pinedo, Phys. Lett. B 436, 19, 1998;
    K. Langanke, G. Martinez-Pinedo, Nucl. Phys. A 673, 481, 2000;
    K. Langanke, G. Martinez-Pinedo, At. Data Nucl. Data Tables 79, 1, 2001.

Introduction — Theoretical description

RPA approach: more suitable for $\begin{cases} \text{the inclusion of forbidden transitions} \\ \text{global calculations of many nuclei} \end{cases}$

- Hybrid model (SMMC/RPA): for nuclei with $Z < 40, N > 40$
  - ✓ The SMMC firstly gives the finite temperature occupation numbers, and then the electron capture rates are calculated within the RPA approach.

    K. Langanke, E. Kolbe, and D. J. Dean, Phys. Rev. C 63, 032801, 2001.
- QRPA based on Nilsson model with separable Gamow-Teller forces: for $A = 18 - 100$

  J. -U. Nabi, and H. V. Klapdor-Kleingrothaus, At. Data Nucl. Data Tables 71, 149, 1999;

  J. -U. Nabi, and H. V. Klapdor-Kleingrothaus, At. Data Nucl. Data Tables 88, 237, 2004.
- QRPA based on Woods-Saxon potential with thermofield dynamics(TFD) formalism

  A. A. Dzhioev, A. I. Vdovin, V. Y. Ponomarev, J. Wambach, K. Langanke, and G. Martínez-Pinedo, Phys. Rev. C 81, 015804, 2010.
- finite temperature RPA based on Skyrme functionals
  - ✓ The self-consistent RPA approach is for the first time introduced to the evaluation of electron capture cross sections.

    N. Paar, G. Colò, E. Khan and D. Vretenar, Phys. Rev. C 80, 055801, 2009.

Introduction — Motivation

Success of covariant density functional

- RMF, RHB : successful for the description of ground-state properties in nuclei all over the periodic table, including those far away from the stability line.

  P. Ring, Prog. Part. Nucl. Phys. 37, 193, 1996.
  D. Vretenar, A. V. Afanasjev, G. A. Lalazissis and P. Ring, Phys. Rep. 409, 101, 2005.

  J. Meng, H. Toki, S. G. Zhou, S. Q. Zhang, W. H. Long, and L. S. Geng, Prog. Part. Nucl. Phys. 57, 470, 2006.

- Relativistic RPA (RRPA): giant resonances, spin isospin resonancs

  Z. Y. Ma, V. Giai Nguyen, A. Wandelt D. Vretenar and P. Ring, Nucl. Phys. A 686, 173, 2001.
  P. Ring, Z. Y. Ma, V. Giai Nguyen, D. Vretenar, A. Wandelt and L. G. Cao, Nucl. Phys. A 694, 249, 2001.
  T. Nikšić, D. Vretenar, and P. Ring, Phys. Rev. C 66, 064302, 2002.
  N. Paar, P. Ring, T. Nikšić and D. Vretenar,Phys. Rev. C 67, 034312, 2003.
  N. Paar, T. Nikšić, D. Vretenar, and P. Ring, Phys. Rev. C 69, 054303, 2004.

  H. Z. Liang, V. Giai Nguyen, and J. Meng, Phys. Rev. Lett. 101, 122502, 2008.

- finite temperature RRPA: new low-lying structure of dipole response

  Y.F. Niu, N. Paar, D. Vretenar and J. Meng, Phys. Lett. B 681, 315, 2009.

In this work

Investigate the stellar electron capture cross sections and rates based on finite temperature RRPA with the inclusion of multipole transitions.

Y.F. Niu, N. Paar, D. Vretenar and J. Meng, Phys.Rev. C 83, 045807, 2011.

PN-RRPA at finite temperature

- Proton-neutron RPA equations at finite temperature

$$\begin{pmatrix} A^J_{pnp'n'} & B^J_{pnp'n'} \\ -B^J_{pnp'n'} & -A^J_{pnp'n'} \end{pmatrix} \begin{pmatrix} X^J_{p'n'} \\ Y^J_{p'n'} \end{pmatrix} = \omega_\nu \begin{pmatrix} X^J_{pn} \\ Y^J_{pn} \end{pmatrix},$$

where A and B are the matrix elements of particle-hole residual interactions:

$$A^J_{pnp'n'} = (\epsilon_p - \epsilon_{\bar{h}})\delta_{pp'}\delta_{nn'} + V^J_{pn'np'}(\tilde{u}_p\tilde{v}_n\tilde{u}_{p'}\tilde{v}_{n'} + \tilde{v}_p\tilde{u}_n\tilde{v}_{p'}\tilde{u}_{n'})(f_{n'} - f_{p'})$$
$$B^J_{pnp'n'} = V^J_{pn'np'}(\tilde{u}_p\tilde{v}_n\tilde{v}_{p'}\tilde{u}_{n'} + \tilde{v}_p\tilde{u}_n\tilde{u}_{p'}\tilde{v}_{n'})(f_{p'} - f_{n'}),$$

- Occupation probability: $f_i = [1 + exp(\frac{\epsilon_i - \mu}{kT})]^{-1}$, and we define

$$\tilde{u}_p = 0, \quad \tilde{v}_p = 1, \quad \tilde{u}_n = 1, \quad \tilde{v}_n = 0, \ \text{when } f_p > f_n \quad (\bar{p}n);$$
$$\tilde{u}_p = 1, \quad \tilde{v}_p = 0, \quad \tilde{u}_n = 0, \quad \tilde{v}_n = 1, \ \text{when } f_p < f_n \quad (p\bar{n}).$$

- Normalization

$$\sum_{pn}[(X^{J\nu}_{pn})^2 - (Y^{J\nu}_{pn})^2]|f_p - f_n| = 1$$

- Transition strength

$$B^{T_-}_{J\nu} = |\sum_{pn}(X^{J\nu}_{pn}\tilde{u}_p\tilde{v}_n + Y^{J\nu}_{pn}\tilde{v}_p\tilde{u}_n)\langle p||T_-||n\rangle|f_n - f_p||^2,$$
$$B^{T_+}_{J\nu} = |\sum_{pn}(X^{J\nu}_{pn}\tilde{v}_p\tilde{u}_n + Y^{J\nu}_{pn}\tilde{u}_p\tilde{v}_n)\langle p||T_+||n\rangle|f_n - f_p||^2.$$

Electron capture rate

- Rate

$$\lambda_{ec} = \frac{1}{\pi^2 \hbar^3} \int_{E_e^0}^{\infty} p_e E_e \sigma_{ec}(E_e) f(E_e, \mu_e, T) dE_e \tag{3}$$

where $E_e^0 = max(|Q_{if}|, m_e c^2)$, $Q_{if} = -E_{\mathrm{RPA}} - \Delta_{np}$, and $p_e = (E_e^2 - m_e^2 c^4)^{1/2}$.
Electron distribution in stellar environment

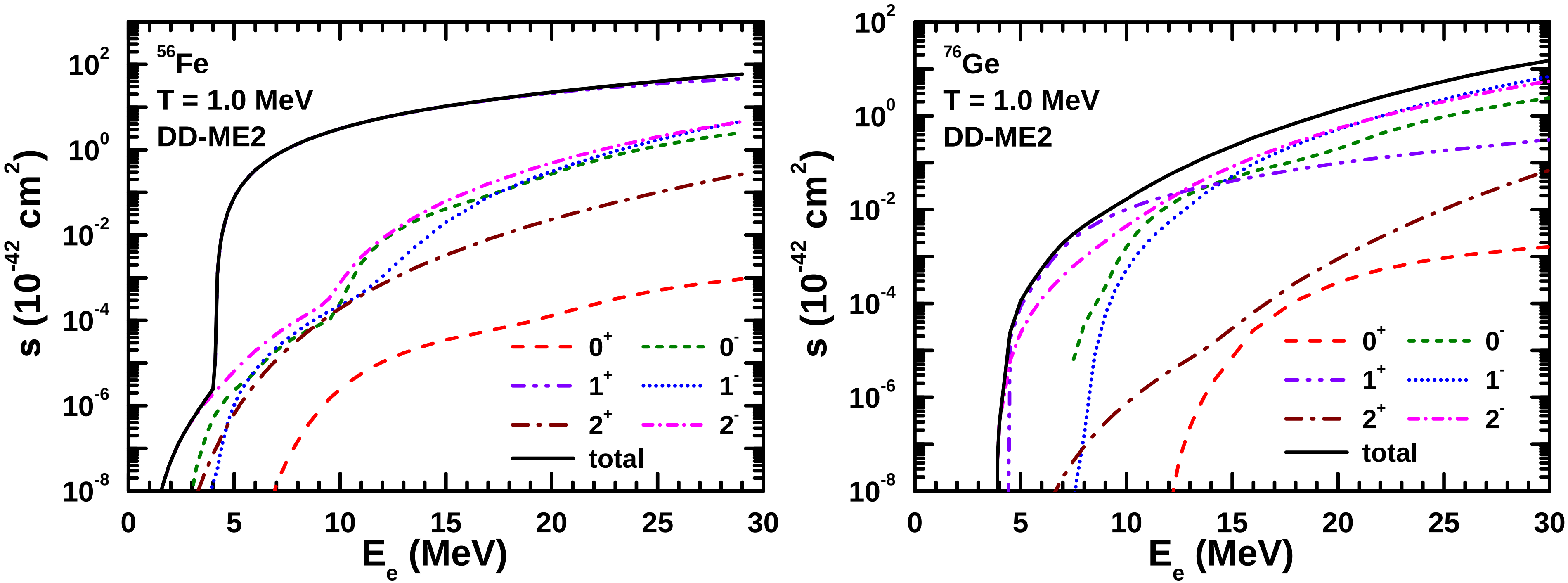
$$f(E_e, \mu_e, T) = \frac{1}{exp(\frac{E_e - \mu_e}{kT}) + 1} \tag{4}$$

where the chemical potential is determined from the density ρ by inverting the relation

$$\rho Y_e = \frac{1}{\pi^2 N_A} \left(\frac{m_e c}{\hbar}\right)^3 \int_0^{\infty} (f_e - f_{e^+}) p^2 dp \tag{5}$$

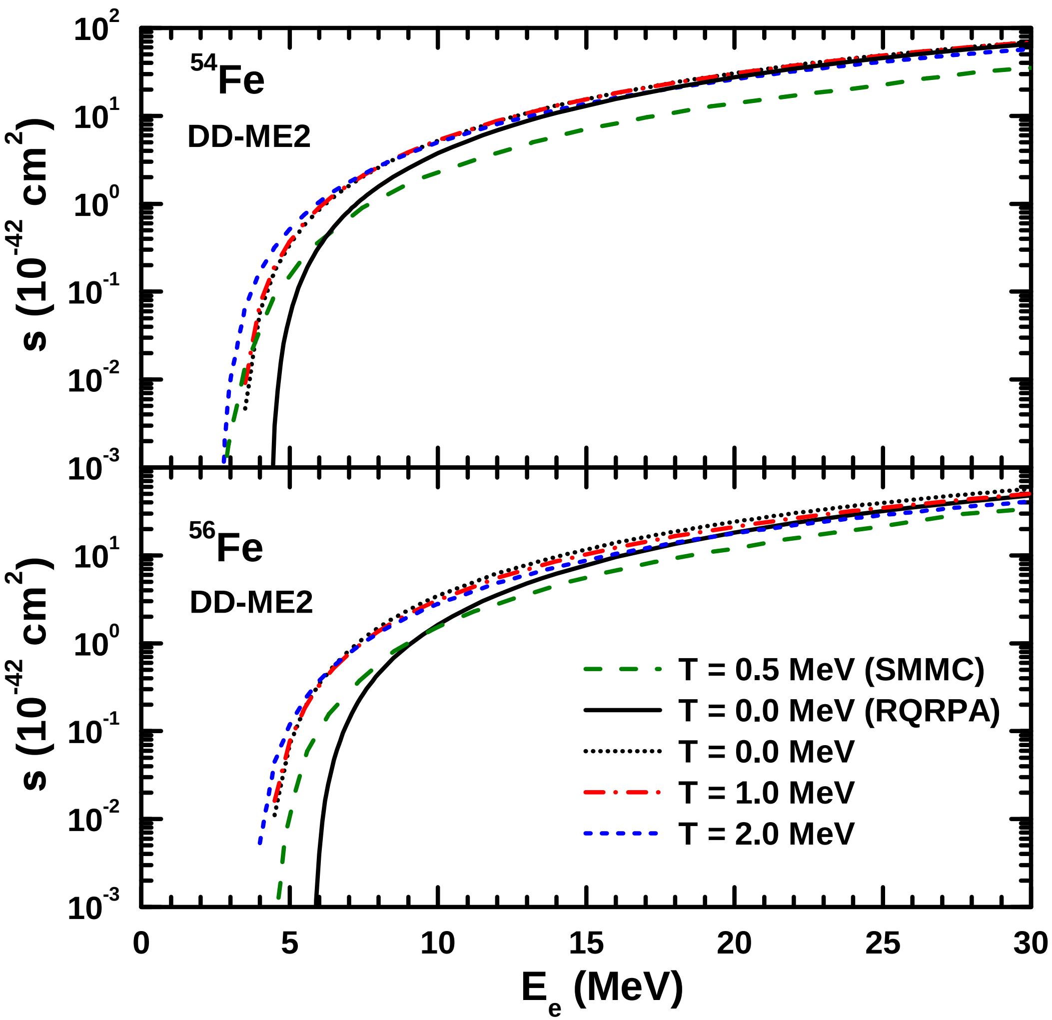
Y_e: the ratio of the number of electrons to the number of baryons; N_A: Avagadro's number; f_e, f_{e^+}: electron and positron distribution.

Contributions from different J^{π} excitations



- For ^{56}Fe, 1^+ (i.e. GT$^+$) gives almost all the contributions to cross section all the way up to $E_e = 30$ MeV. In the other components of excitations, the first forbidden transitions $(0^-, 1^-, 2^-)$ are more important.
- For neutron rich nucleus ^{76}Ge, the first forbidden transitions start to give more contributions than 1^+ from $E_e \backsimeq 12$ MeV.

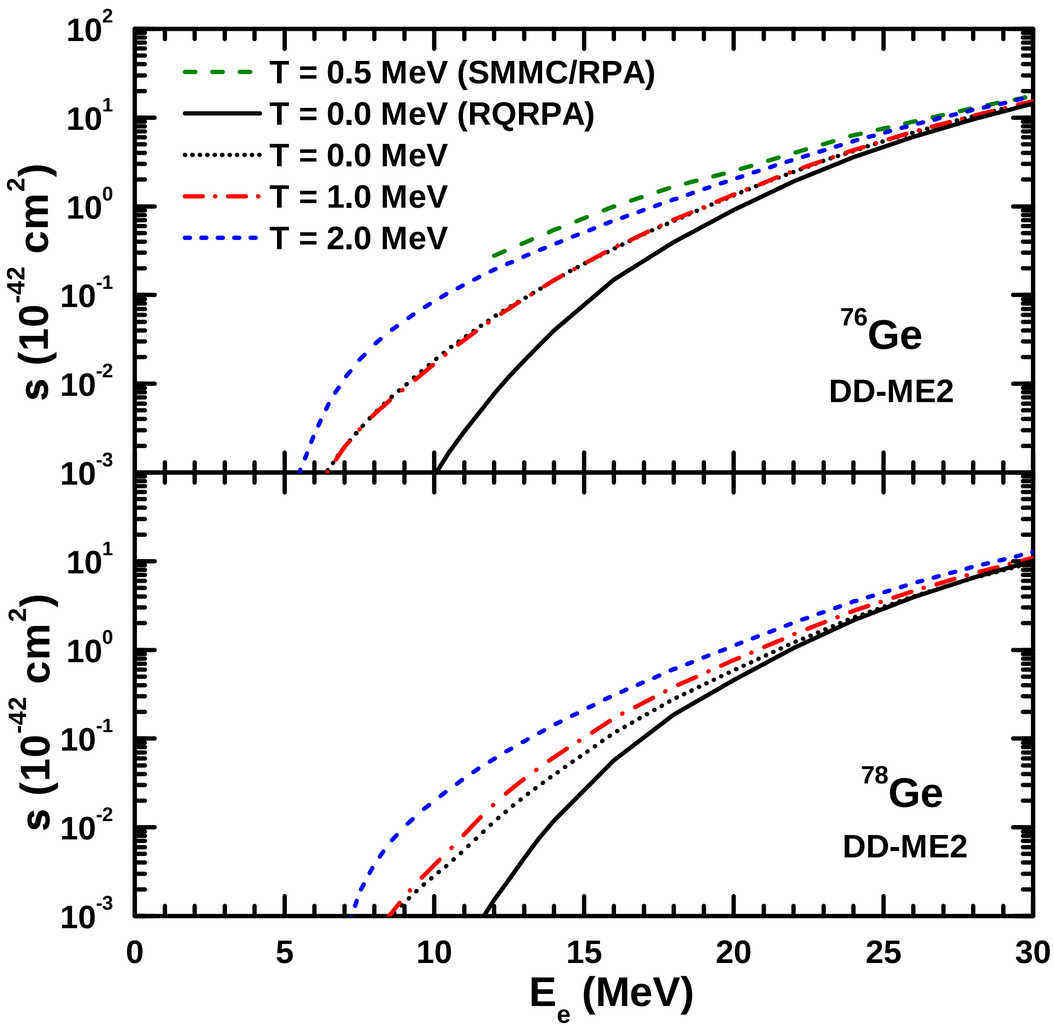
Electron capture cross section



$$\sigma_{ec} \propto E_\nu^2$$
$$E_\nu = E_e - E_{\mathrm{RPA}} - \Delta_{NP}$$

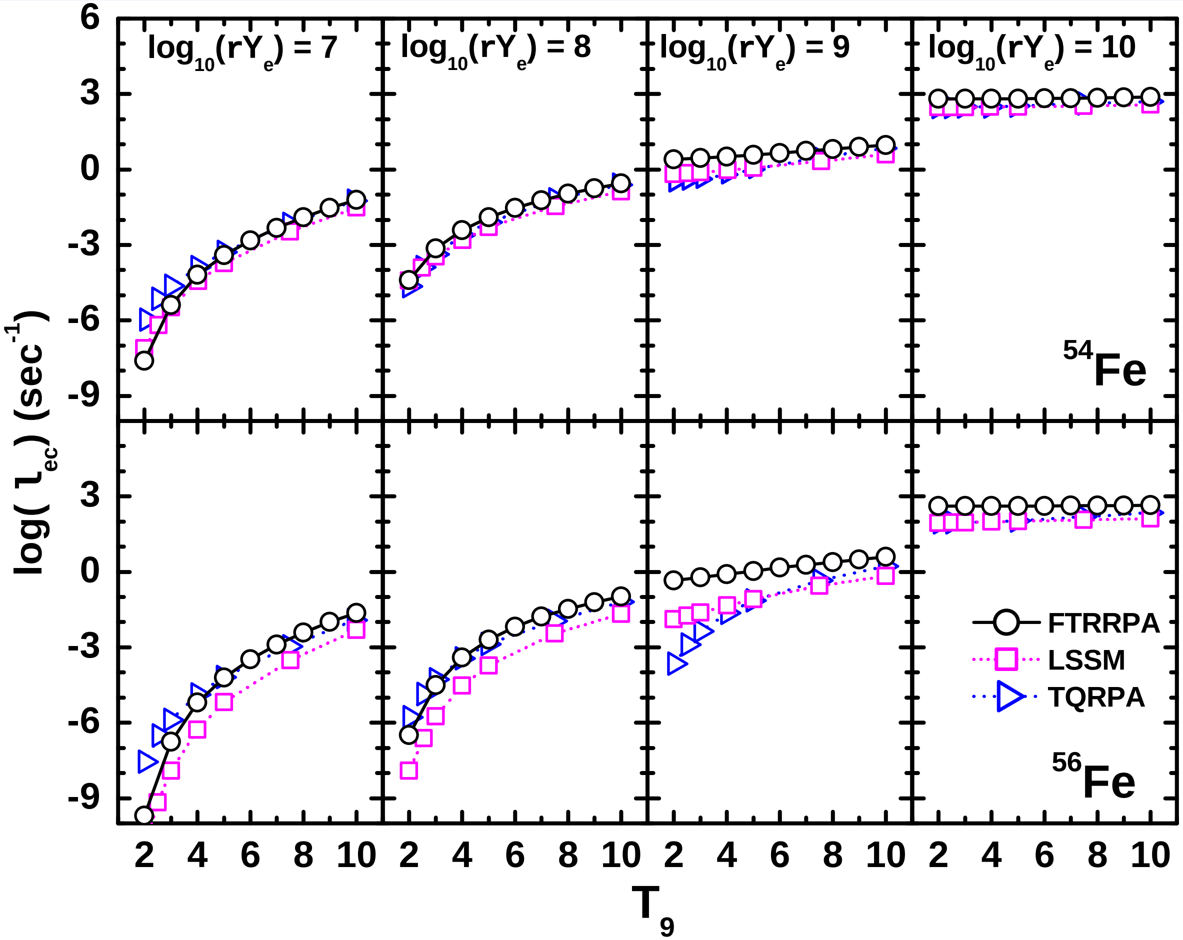
- Threshold energy for electron capture is lowered with increased temperature.
- Cross section is less dependent on temperature at high electron energies.
- Threshold energies increase and cross sections decrease with neutron number.
- difference with SMMC:
  GT distribution

Electron capture cross section



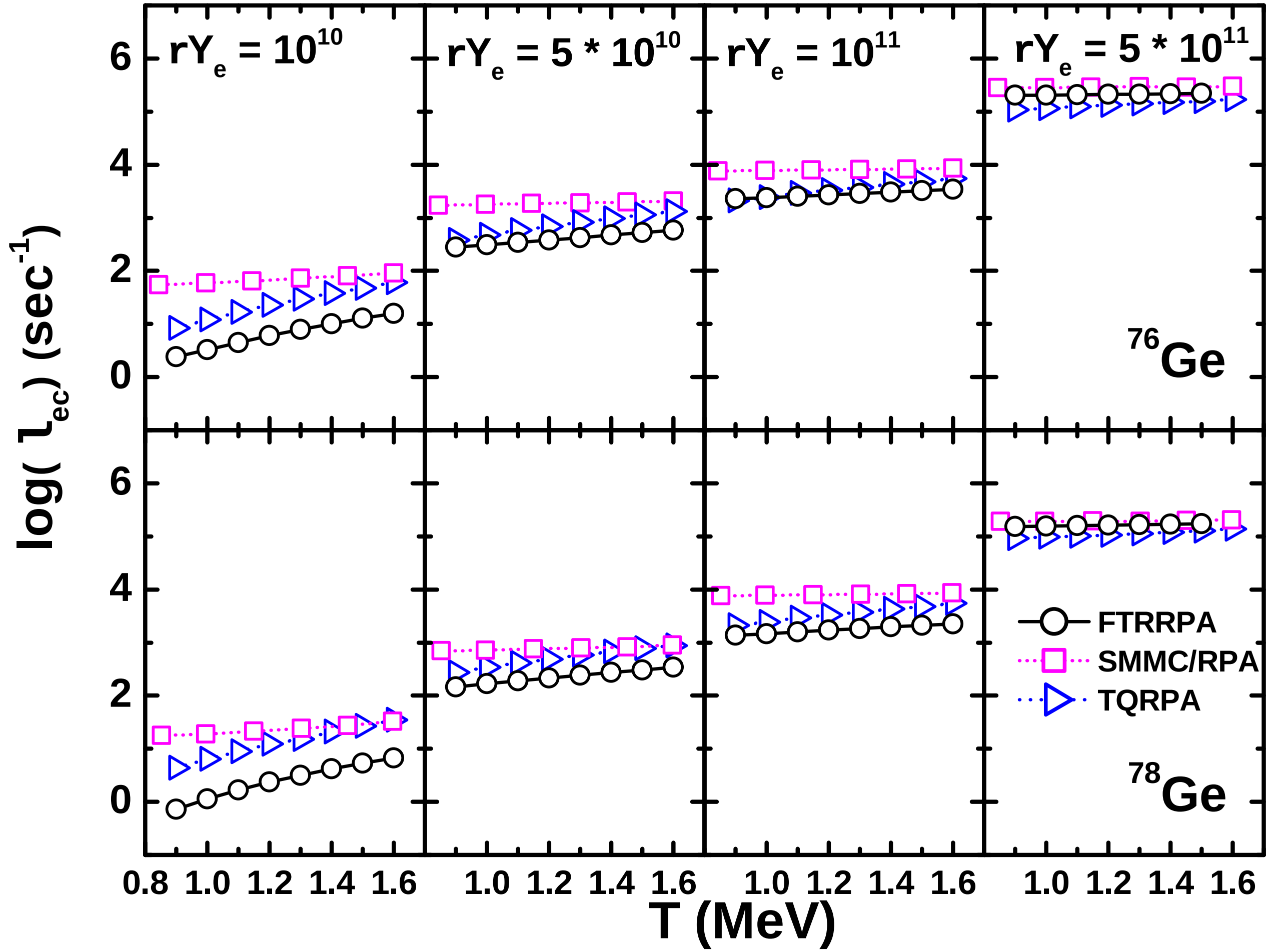
- Cross sections are reduced by an order of magnitude compared to irons, but similar evolution with temperature is found.
- strong temperature dependence at $E_e \leq 12$ MeV.
- SMMC/RPA predicts larger cross sections due to the strong configuration mixing.

Electron capture rate



- Electron capture rates increase with temperature and the electron densities.
  - In high electron densities, it increases more slowly.
- similar trend of temperature dependence as shell model
  - better agreement in ^{54}Fe than ^{56}Fe
  - $\rho Y_e = 10^9$ g/cm^3: $\lambda_e \sim 5$ MeV, close to threshold energy $\Rightarrow$ sensitive to detailed GT distribution $\Rightarrow$ larger discrepancy

Electron capture rate



- similar behavior with temperature and electron density as Fe
- more similar temperature dependence as TQRPA
  - lower densities: strong temperature dependence. $\lambda_e \sim 11$ MeV, dominated by GT
  - larger densities: less sensitive to temperature. $\lambda_e \sim 23$ MeV, dominated by forbidden transitions

Summary & Perspectives

The electron capture cross sections and rates in stellar environment, including multipole excitations, are calculated based on self-consistent finite temperature PN-RRPA.

- In the calculation of cross sections, the GT^+ transitions provide major contribution for ^{54,56}Fe, whereas for ^{76,78}Ge forbidden transitions play an important role at $E_e > 10$ MeV.
- The principal effect of increasing temperature is the lowering of the electron-capture threshold energy. For ^{76,78}Ge the cross sections in the low-energy region are very sensitive to temperature.
- In the calculation of capture rates, for ^{54,56}Fe FTRRPA displays a similar trend as shell model. For ^{76,78}Ge, the temperature dependence in FTRRPA is very close to TQRPA, whereas the dependence is much weaker in the hybrid model.

Perspectives:

- for open-shell nuclei at very low temperatures: pairing correlations
- the inclusion of higher-order correlations beyond the RPA level

Acknowledgements

Collaborators:

Jie Meng:	School of Physics, Peking University, China
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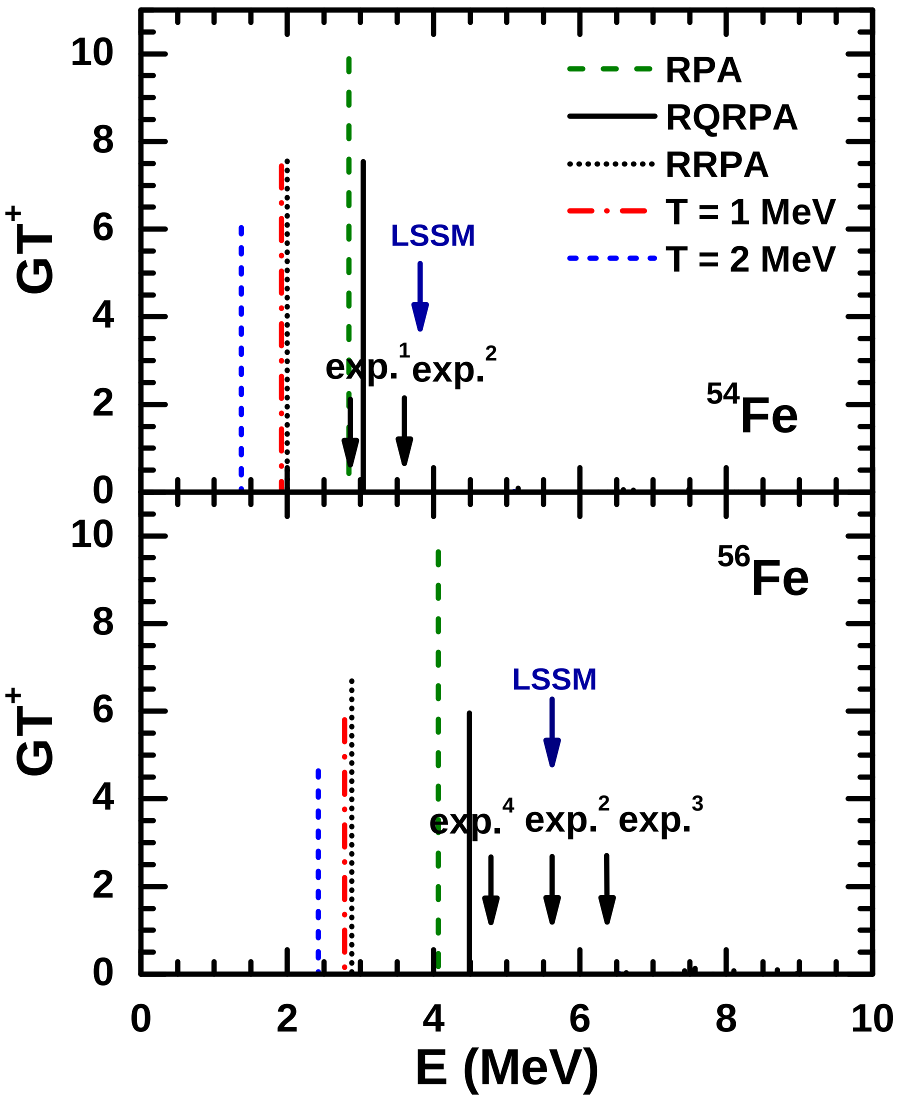
Dario Vretenar:	Physics Department, University of Zagreb, Croatia

THANK YOU !

Evolution of GT+ with temperature



- Pairing correlations shift the transition to higher energy, because additional energy is needed to break a proton pair.
- From T=0 (RQRPA) to T=1, energy decreases due to pairing collapse.
- From T=1 to T=2, energy decreases due to softening of the repulsive residual interaction.
- Transition strength becomes weaker with increasing temperature or with pairing by the partial occupation factors.

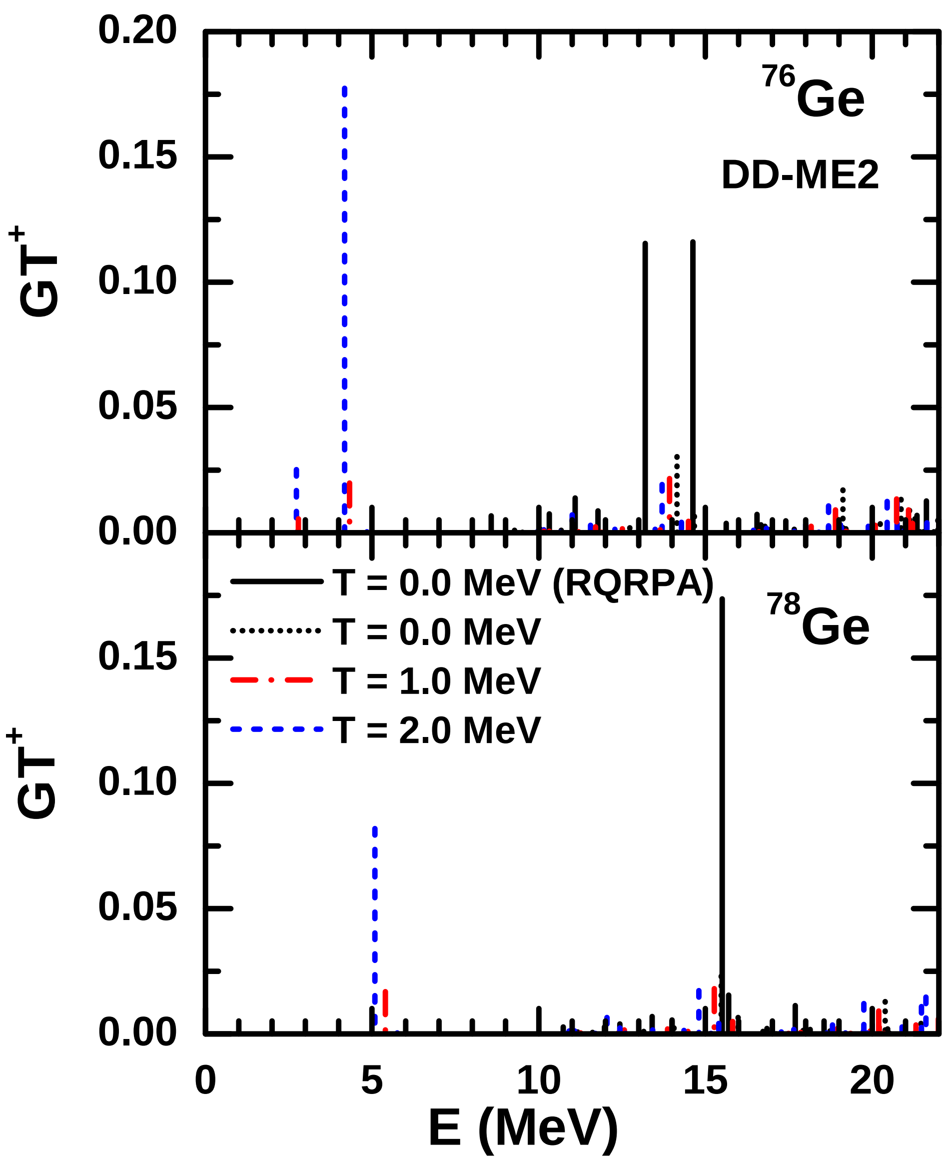
[1] M. C. Vetterli, et. al. Phys. Rev. C 40, 559, 1989.

[2] T. Rönnqvist, et. al. Nucl. Phys. A 563, 225, 1993.

[3] S. El-Kateb, et. al. Phys. Rev. C 49, 3128, 1994.

[4] D. Frekers, Nucl. Phys. A 752, 580c, 2005.

Evolution of GT$^+$ with temperature



- Unblocking mechanisms:
  pairing correlations & thermal excitations.
- From T=0 (RQRPA) to T=1:
  - energy decreases much
    T=0:
    $$\sqrt{(\epsilon_p-\lambda_p)^2+\Delta_p^2}+\sqrt{(\epsilon_n-\lambda_n)^2+\Delta_n^2}$$
    T=1: $\epsilon_n-\epsilon_p$.
  - strength decreases much
    pairing correlation $\Rightarrow$ more diffused fermi surface
- From T=1 to T=2:
  energy decreases a little while strength is enhanced a lot.

Electron capture cross section

The electron capture on a nucleus ${}^{A}_{Z}X_N$

$$e^- + {}^{A}_{Z}X_N \rightarrow {}^{A}_{Z-1}X^*_{N+1} + \nu_e \tag{6}$$

- Cross section for a transition between initial state $|i\rangle$ and a final state $|f\rangle$ followed from Fermi's golden rule reads:

$$\frac{d\sigma}{d\Omega} = 2\pi|\langle f|\hat{H}_W|i\rangle|^2 V\frac{E_\nu^2 dE_\nu}{(2\pi)^3}\delta(W_f - W_i)/\frac{1}{V}, \tag{7}$$

  where $V\frac{E_\nu^2 dE_\nu}{(2\pi)^3}$ is the number of neutrino states in the interval $E_\nu \sim E_\nu + dE_\nu$, and electron flux is $1/V$. $\delta(W_f - W_i)$ means the energy conservation.
- Average the initial states and sum over all the final states for a specific nuclear excitation state f:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{1}{2J_i+1}\sum_{M_i}\frac{1}{2}\sum_{\text{lepton spins}}\sum_{M_f}\int dE_\nu 2\pi|\langle f|\hat{H}_W|i\rangle|^2 V\frac{E_\nu^2}{(2\pi)^3}\delta(W_f - W_i)/\frac{1}{V}\\ &= \frac{1}{(2\pi)^2}V^2E_\nu^2\frac{1}{2}\sum_{\text{lepton spins}}\frac{1}{2J_i+1}\sum_{M_iM_f}|\langle f|\hat{H}_W|i\rangle|^2.\end{aligned} \tag{8}$$

Electron capture cross section

- The Hamiltonian of the weak interaction H_W is expressed in the standard current-current form, i.e., in terms of the nuclear $\mathcal{J}_\lambda$ and lepton j_λ currents.

$$H_W = -\frac{G}{\sqrt{2}} \int d^3x \mathcal{J}_\lambda(\mathbf{x}) j_\lambda(\mathbf{x}) \tag{9}$$

  G: weak coupling constant.

- Denoting the appropriate leptonic matrix element by $l_\mu e^{-i\mathbf{q}\cdot\mathbf{x}}$, the resulting transition matrix element reads

$$\begin{aligned} \langle f|\hat{H}_W|i\rangle &= -\frac{G}{\sqrt{2}} l_\mu \int d\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} \langle f|\mathcal{J}_\mu(\mathbf{x})|i\rangle \\ &= -\frac{G}{\sqrt{2}} \int d\mathbf{x} e^{-i\mathbf{q}\cdot\mathbf{x}} [\mathbf{l}\cdot\boldsymbol{\mathcal{J}}(\mathbf{x})_{fi} - l_0 \mathcal{J}_0(\mathbf{x})_{fi}]. \end{aligned} \tag{10}$$

  $\mathbf{q} = \mathbf{p}_\nu - \mathbf{p}_l$: the momentum transfer.

Electron capture cross section

Making use of the expansion

$$e^{i\ \mathbf{q}\cdot\mathbf{x}} = \sum_{J=0}^{\infty}[4\pi(2J+1)]^{1/2} i^J j_J(\kappa x) Y_{J0}(\Omega_x), \tag{11}$$

$$\mathbf{e}_{q\lambda} e^{i\mathbf{q}\cdot\mathbf{x}} = -\frac{i}{\kappa}\sum_{J=0}^{\infty}[4\pi(2J+1)]^{1/2} i^J \nabla(j_J(\kappa x) Y_{J0}(\Omega_x)), \text{ for } \lambda = 0, \tag{12}$$

$$= -\sum_{J\geq 1}^{\infty}[2\pi(2J+1)]^{1/2} i^J [\lambda j_J(\kappa x)\mathcal{Y}^{\lambda}_{JJ1} + \frac{1}{\kappa}\nabla\times(j_J(kx)\mathcal{Y}^{\lambda}_{JJ1})], \text{ for } \lambda = \pm 1, \tag{13}$$

where $\mathcal{Y}^{M}_{Jl1} = \sum_{m\lambda}\langle lm1\lambda|l1JM\rangle Y_{lm}(\theta,\phi)\mathbf{e}_{\lambda}$, the transition matrix elements could become

$$\langle f|\hat{H}_W|i\rangle = -\frac{G}{\sqrt{2}}\langle f|\{-\sum_{\lambda=\pm 1} l_\lambda \sum_{J\geq 1}^{\infty}[2\pi(2J+1)]^{1/2}(-i)^J[\lambda\hat{\mathcal{T}}^{mag}_{J-\lambda}(\kappa) + \hat{\mathcal{T}}^{el}_{J-\lambda}(\kappa)]$$

$$+\sum_{J=0}^{\infty}[4\pi(2J+1)]^{1/2}(-i)^J[l_3\hat{\mathcal{L}}_{J0}(\kappa) - l_0\hat{\mathcal{M}}_{J0}(\kappa)]\}|i\rangle, \tag{14}$$

Electron capture cross section

where the multipole operators are defined by

$$\hat{\mathcal{M}}_{JM}(\kappa) = \hat{M}_{JM} + \hat{M}^5_{JM} = \int d\mathbf{x}[j_J(\kappa x) Y_{JM}(\Omega_x)]\hat{\mathcal{J}}_0(\mathbf{x}), \tag{15}$$

$$\hat{\mathcal{L}}_{JM}(\kappa) = \hat{L}_{JM} + \hat{L}^5_{JM} = \frac{i}{\kappa}\int d\mathbf{x}[\boldsymbol{\nabla}(j_J(\kappa x) Y_{JM}(\Omega_x))] \cdot \hat{\boldsymbol{\mathcal{J}}}(\mathbf{x}), \tag{16}$$

$$\hat{\mathcal{T}}^{el}_{JM}(\kappa) = \hat{T}^{el}_{JM} + \hat{T}^{el5}_{JM} = \frac{1}{\kappa}\int d\mathbf{x}[\boldsymbol{\nabla} \times (j_J(\kappa x)\boldsymbol{\mathcal{Y}}^M_{JJ1})] \cdot \hat{\boldsymbol{\mathcal{J}}}(\mathbf{x}), \tag{17}$$

$$\hat{\mathcal{T}}^{mag}_{JM}(\kappa) = \hat{T}^{mag}_{JM} + \hat{T}^{mag5}_{JM} = \int d\mathbf{x}[(j_J(\kappa x)\boldsymbol{\mathcal{Y}}^M_{JJ1})] \cdot \hat{\boldsymbol{\mathcal{J}}}(\mathbf{x}). \tag{18}$$

Electron capture cross section

Making use of the Wigner-Eckart theorem

$$\langle J_f M_f | \mathcal{T}_{JM} | J_i M_i \rangle = (-)^{J_f - M_f} \begin{pmatrix} J_f & J & J_i \\ -M_f & M & M_i \end{pmatrix} \langle J_f || \mathcal{T}_J || J_i \rangle \tag{19}$$

and the orthogonality relation of $3j$ coefficients

$$\frac{1}{2J_i+1} \sum_{M_f} \sum_{M_i} \begin{pmatrix} J_f & J & J_i \\ -M_f & M & M_i \end{pmatrix} \begin{pmatrix} J_f & J' & J_i \\ -M_f & M' & M_i \end{pmatrix} = \delta_{JJ'} \delta_{MM'} \frac{1}{2J+1} \frac{1}{2J_i+1}, \tag{20}$$

we could get

$$\begin{aligned} & \frac{1}{2J_i+1} \sum_{M_f} \sum_{M_i} |\langle f | \hat{H}_W | i \rangle|^2 \\ = & \frac{G^2}{2} \frac{1}{2J_i+1} \{ \sum_{\lambda=\pm1} l_\lambda l_\lambda^* \sum_{J\geq1} 2\pi |\langle J_f || \lambda \mathcal{T}_J^{mag} + \mathcal{T}_J^{el} || J_i \rangle|^2 \\ & + \sum_{J\geq0} 4\pi [l_3 l_3^* |\langle J_f || \mathcal{L}_J || J_i \rangle|^2 + l_0 l_0^* |\langle J_f || \mathcal{M}_J || J_i \rangle|^2 - 2\mathrm{Re} l_3 l_0^* \langle J_f || \mathcal{L}_J || J_i \rangle \langle J_f || \mathcal{M}_J || J_i \rangle^*] \}. \end{aligned} \tag{21}$$

Electron capture cross section

Until now, the electron capture cross section has the form

$$\frac{d\sigma}{d\Omega} = \frac{1}{(2\pi)^2}V^2E_\nu^2\frac{1}{2}\sum_{\text{lepton spins}}\frac{1}{2J_i+1}\sum_{M_iM_f}|\langle f|\hat{H}_W|i\rangle|^2$$
$$= \frac{1}{(2\pi)^2}V^2E_\nu^2\frac{1}{2}\sum_{\text{lepton spins}}\frac{G^2}{2}\frac{1}{2J_i+1}\{\sum_{\lambda=\pm1}l_\lambda l_\lambda^*\sum_{J\geq1}2\pi|\langle J_f||\lambda\mathcal{T}_J^{mag}+\mathcal{T}_J^{el}||J_i\rangle|^2$$
$$+\sum_{J\geq0}4\pi[l_3l_3^*|\langle J_f||\mathcal{L}_J||J_i\rangle|^2+l_0l_0^*|\langle J_f||\mathcal{M}_J||J_i\rangle|^2-2\mathrm{Re}l_3l_0^*\langle J_f||\mathcal{L}_J||J_i\rangle\langle J_f||\mathcal{M}_J||J_i\rangle^*]\}.$$

As the summation of leptonic matrix element on lepton spins could be related to the unit vectors of neutrino momentum $\hat{\boldsymbol{\nu}} = \mathbf{p}_\nu/|\mathbf{p}_\nu|$, transfer momentum $\hat{\mathbf{q}} = \mathbf{q}/|\mathbf{q}|$, and $\boldsymbol{\beta} = \mathbf{p}_e/E_e$, the electron capture cross section finally becomes

$$\frac{d\sigma}{d\Omega} = \frac{G^2}{2\pi}\frac{E_\nu^2}{2J_i+1}\{\sum_{J\geq0}\{(1-\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta}+2(\hat{\boldsymbol{\nu}}\cdot\hat{\mathbf{q}})(\boldsymbol{\beta}\cdot\hat{\mathbf{q}}))|\langle J_f||\mathcal{L}_J||J_i\rangle|^2$$
$$+(1+\hat{\boldsymbol{\nu}}\cdot\boldsymbol{\beta})|\langle J_f||\mathcal{M}_J||J_i\rangle|^2-2\hat{\mathbf{q}}\cdot(\hat{\boldsymbol{\nu}}+\boldsymbol{\beta})\mathrm{Re}\langle J_f||\mathcal{L}_J||J_i\rangle\langle J_f||\mathcal{M}_J||J_i\rangle^*\}$$
$$+\sum_{J\geq1}\{(1-(\hat{\boldsymbol{\nu}}\cdot\hat{\mathbf{q}})(\boldsymbol{\beta}\cdot\hat{\mathbf{q}}))[|\langle J_f||\mathcal{T}_J^{mag}||J_i\rangle|^2+|\langle J_f||\mathcal{T}_J^{el}||J_i\rangle|^2]$$
$$-2\hat{\mathbf{q}}\cdot(\hat{\boldsymbol{\nu}}-\boldsymbol{\beta})\mathrm{Re}\langle J_f||\mathcal{T}_J^{mag}||J_i\rangle\langle J_f||\mathcal{T}_J^{el}||J_i\rangle^*\}\}. \quad (22)$$

Electron capture cross section

◇ Allowed Processes: the long-wave limit where $\kappa = |\mathbf{q}| \to 0$.
The only surviving multipoles in this limit are

$$\mathcal{T}^{el}_{1M} = \frac{i}{\sqrt{6\pi}} F_A \sum_{i=1}^{A} \tau_{\pm}(i)\boldsymbol{\sigma}(i); \qquad \text{Gamow-Teller} \tag{23}$$

$$\mathcal{M}_{00} = \frac{1}{\sqrt{4\pi}} F_1 \sum_{i=1}^{A} \tau_{\pm}(i); \qquad \text{Fermi} \tag{24}$$

Remarks

- These operators give rise to the allowed weak transitions in the traditional picture of the nucleus.
- The operators and transitions they give rise to are known as Gamow-Teller and Fermi respectively.