η and η' mesons in the Dyson-Schwinger approach with the generalized Witten-Veneziano relations[a]

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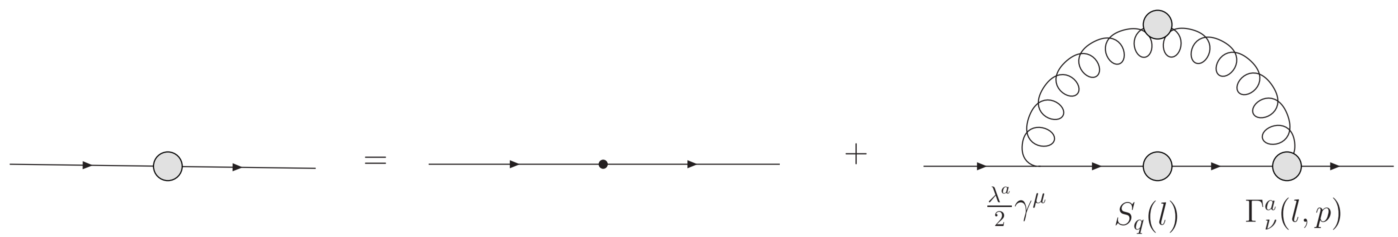
Dyson-Schwinger approach to quark-hadron physics

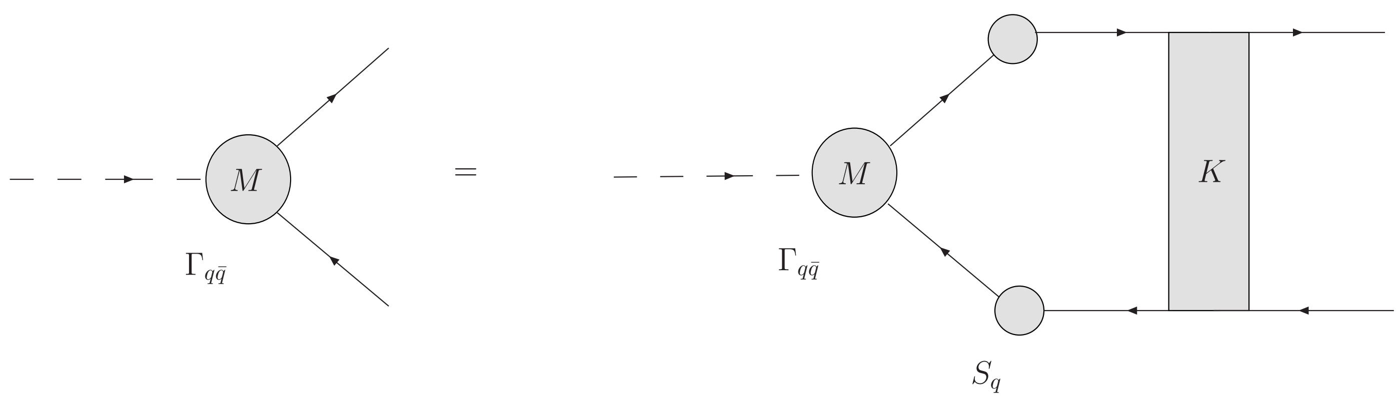
- = the bound state approach which is nopertubative, covariant and chirally well behaved (e.g., GMOR relation: $\lim_{\widetilde{m}_q \to 0} \; M_{q\bar{q}}^2 / 2\widetilde{m}_q = -\langle \bar{q}q \rangle / f_\pi^2$)
- a) direct contact with QCD through *ab initio* calculations
- b) phenomenological modeling of hadrons as quark bound states (e.g., here)
- coupled system of integral equations for Green functions of QCD
- ... but ... equation for n-point function calls (n+1)-point function ... $\longrightarrow$ cannot solve in full the growing tower of DS equations
- $\longrightarrow$ various degrees of truncations, approximations and modeling is unavoidable (more so in phenomenological modeling of hadrons, as here)

Dyson-Schwinger approach to quark-hadron physics

- Gap equation for propagator S_q of dressed quark q



- Homogeneous Bethe-Salpeter (BS) equation for a Meson $q\bar{q}$ bound state vertex $\Gamma_{q\bar{q}}$



Gap and BS equations in ladder truncation

$$S_q(p)^{-1} = i\gamma\cdot p + \widetilde{m}_q + \frac{4}{3}\int\frac{d^4\ell}{(2\pi)^4}\, g^2 G^{\text{eff}}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell)\gamma_\nu$$

$$\to S_q(p) = \frac{1}{i\not p A_q(p^2) + B_q(p^2)} = \frac{-i\not p A_q(p^2) + B_q(p^2)}{p^2 A_q(p^2)^2 + B_q(p^2)^2} = \frac{1}{A_q(p^2)}\frac{-i\not p + m_q(p^2)}{p^2 + m_q(p^2)^2}$$

$$\Gamma_{q\bar{q}'}(p,P) = -\frac{4}{3}\int\frac{d^4\ell}{(2\pi)^4} g^2 G^{\text{eff}}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell+\frac{P}{2})\Gamma_{q\bar{q}'}(\ell,P)S_q(\ell-\frac{P}{2})\gamma_\nu$$

- Euclidean space: $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$, $a\cdot b = \sum_{i=1}^4 a_i b_i$
- P is the total momentum, $M^2 = -P^2$ meson mass^2
- $G^{\text{eff}}_{\mu\nu}(k)$ an “effective gluon propagator” - modeled !

From the gap and BS equations ...

- solutions of the gap equation → the dressed quark mass function

$$m_q(p^2) = \frac{B_q(p^2)}{A_q(p^2)}$$

- propagator solutions $A_q(p^2)$ and $B_q(p^2)$ pertain to confined quarks if

$$m_q^2(p^2) \neq -p^2 \quad \text{for} \;\; \text{real} \quad p^2$$

- The BS solutions $\Gamma_{q\bar{q}'}$ enable the calculation of the properties of $q\bar{q}$ bound states, such as the decay constants of pseudoscalar mesons:

$$\begin{aligned} f_{PS}\, P_\mu &= \langle 0|\bar{q}\, \frac{\lambda^{PS}}{2}\, \gamma_\mu\, \gamma_5\, q|\Phi_{PS}(P)\rangle \\ \longrightarrow \quad f_\pi P_\mu &= N_c\, \mathrm{tr_s} \int \frac{d^4\ell}{(2\pi)^4}\, \gamma_5 \gamma_\mu\, S(\ell + P/2)\, \Gamma_\pi(\ell; P)\, S(\ell - P/2) \end{aligned}$$

Renormalization-group improved interactions

$$\text{Landau gauge gluon propagator}: \quad g^2 G^{\text{eff}}_{\mu\nu}(k) = G(-k^2)(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}),$$

$$G(Q^2) \equiv 4\pi \frac{\alpha_s^{\text{eff}}(Q^2)}{Q^2} = G_{\mathsf{UV}}(Q^2) + G_{\mathsf{IR}}(Q^2), \qquad Q^2 \equiv -k^2 \; .$$

$$G_{\mathsf{UV}}(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \approx \frac{4\pi^2 d}{Q^2 \ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})}\{1 + b \frac{\ln[\ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})]}{\ln(x_0 + \frac{Q^2}{\Lambda^2_{\mathsf{QCD}}})}\} \; ,$$

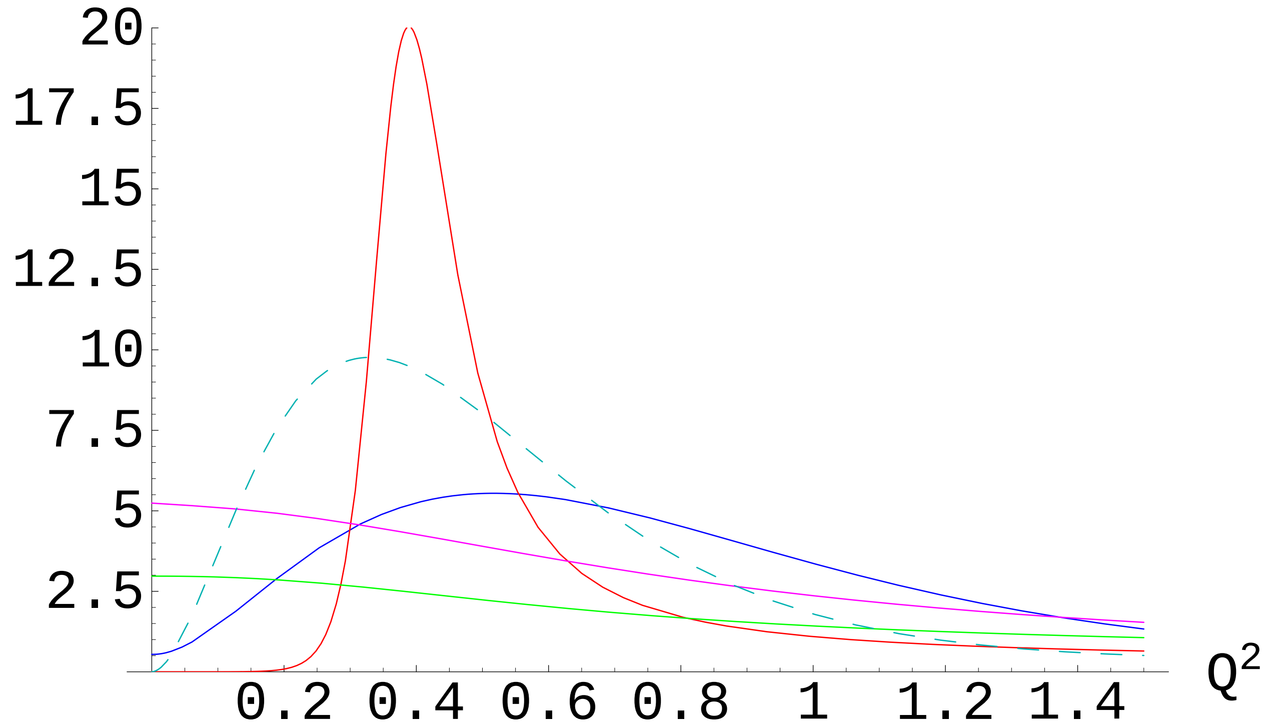
- but modeled non-perturbative part, e.g., Jain & Munczek:

$$G_{\mathsf{IR}}(Q^2) = G_{\mathsf{non\text{-}pert}}(Q^2) = 4\pi^2 \, a \, Q^2 \, \exp(-\mu Q^2) \quad (\text{similar}: \;\; \text{Maris, Roberts...})$$

- or, the dressed propagator with dim. 2 gluon condensate $\langle A^2 \rangle$-induced dynamical gluon mass (Kekez & Klabučar):

$$G(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \left(\frac{Q^2}{Q^2 - M^2_{\mathsf{gluon}} + \frac{c_{\mathsf{ghost}}}{Q^2}} \right)^2 \frac{Q^2}{Q^2 + M^2_{\mathsf{gluon}} + \frac{c_{\mathsf{gluon}}}{Q^2}} \; .$$

Some effective strong couplings $\alpha_s^{\mathrm{eff}}(Q^2) \equiv Q^2\, G(Q^2)/4\pi$



- Blue = Munczek & Jain model. Red = K & K propagator with $\langle A^2 \rangle$-induced dynamical gluon mass. Green = Alkofer. Magenta = Bloch. Turquoise dashed: Maris, Roberts & Tandy model.
  Important: integrated IR strength must be sufficient for **DChSB**!

Separable model

- Calculations simplify with the separable Ansatz for $G^{\text{eff}}_{\mu\nu}$:

$$G^{\text{eff}}_{\mu\nu}(p-q) \rightarrow \delta_{\mu\nu}\, G(p^2, q^2, p\cdot q)$$

$$G(p^2, q^2, p\cdot q) = D_0\; f_0(p^2) f_0(q^2) + D_1\; f_1(p^2)(p\cdot q) f_1(q^2)$$

- two strength parameters D_0, D_1, and corresponding form factors $f_i(p^2)$. In the separable model, gap equation yields

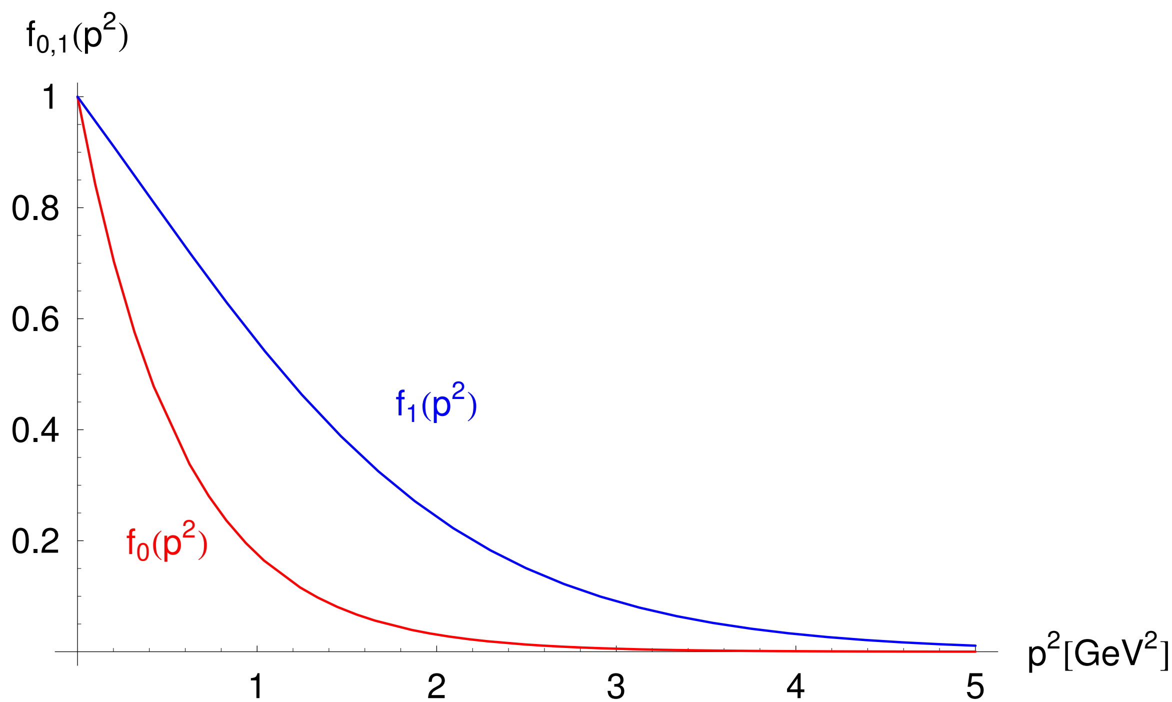
$$\begin{aligned} B_f(p^2) &= \widetilde{m}_f + \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{B_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)} \\ \left[A_f(p^2) - 1\right] p^2 &= \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{(p\cdot q) A_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)}\,. \end{aligned}$$

- This gives $B_f(p^2) = \widetilde{m}_f + b_f\; f_0(p^2)$ and $A_f(p^2) = 1 + a_f\; f_1(p^2)$, reducing to nonlinear equations for constants b_f and a_f.

A simple choice for ‘interaction form factors’ of the separable model:

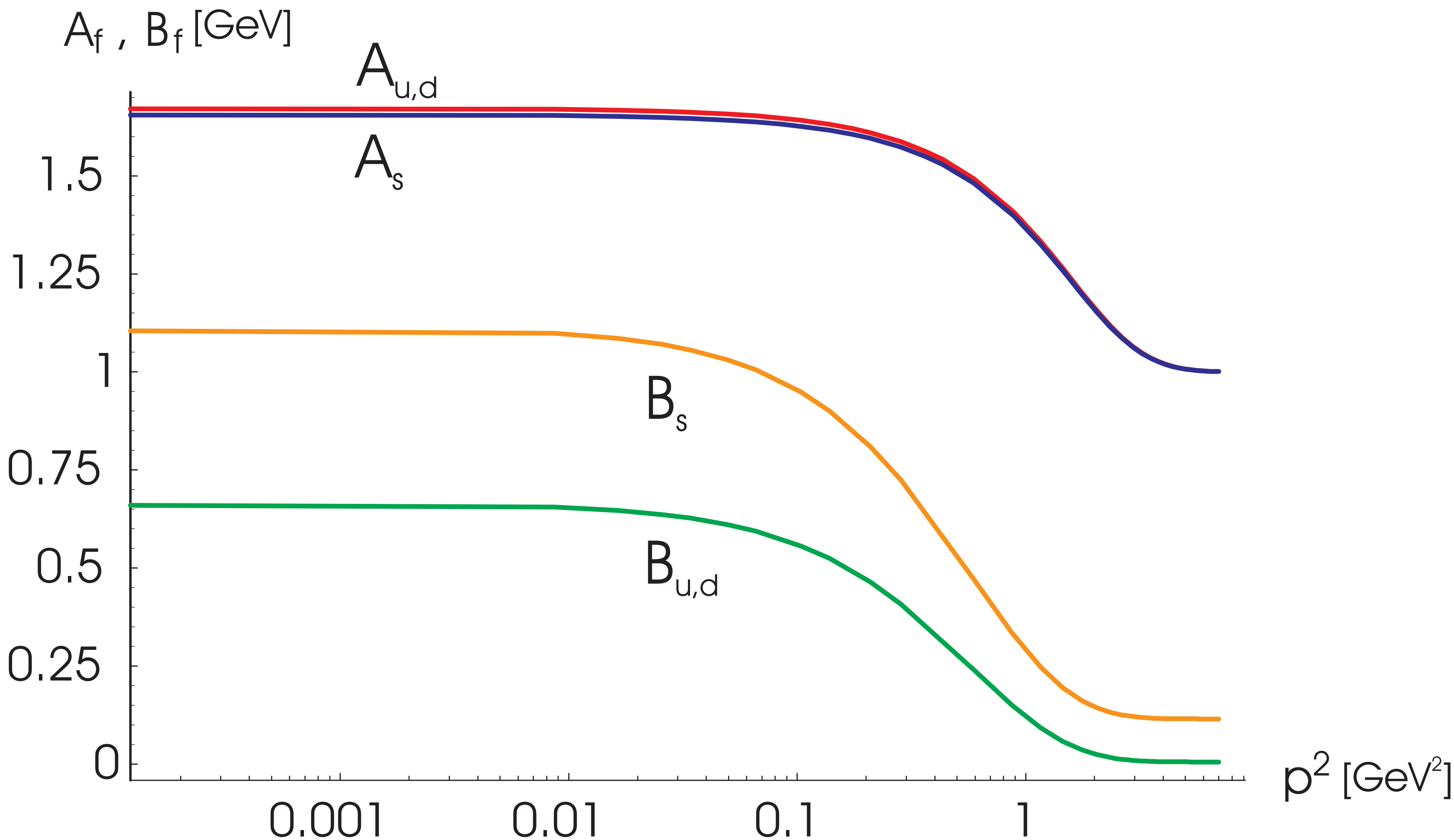
- $f_0(p^2) = \exp(-p^2/\Lambda_0^2)$
- $f_1(p^2) = [1 + \exp(-p_0^2/\Lambda_1^2)]/[1 + \exp((p^2 - p_0^2))/\Lambda_1^2]$

  gives good description of pseudoscalar properties if the interaction is strong enough for realistic DChSB, when $m_{u,d}(p^2 \sim small) \sim$ the typical constituent quark mass scale $\sim m_\rho/2 \sim m_N/3$.



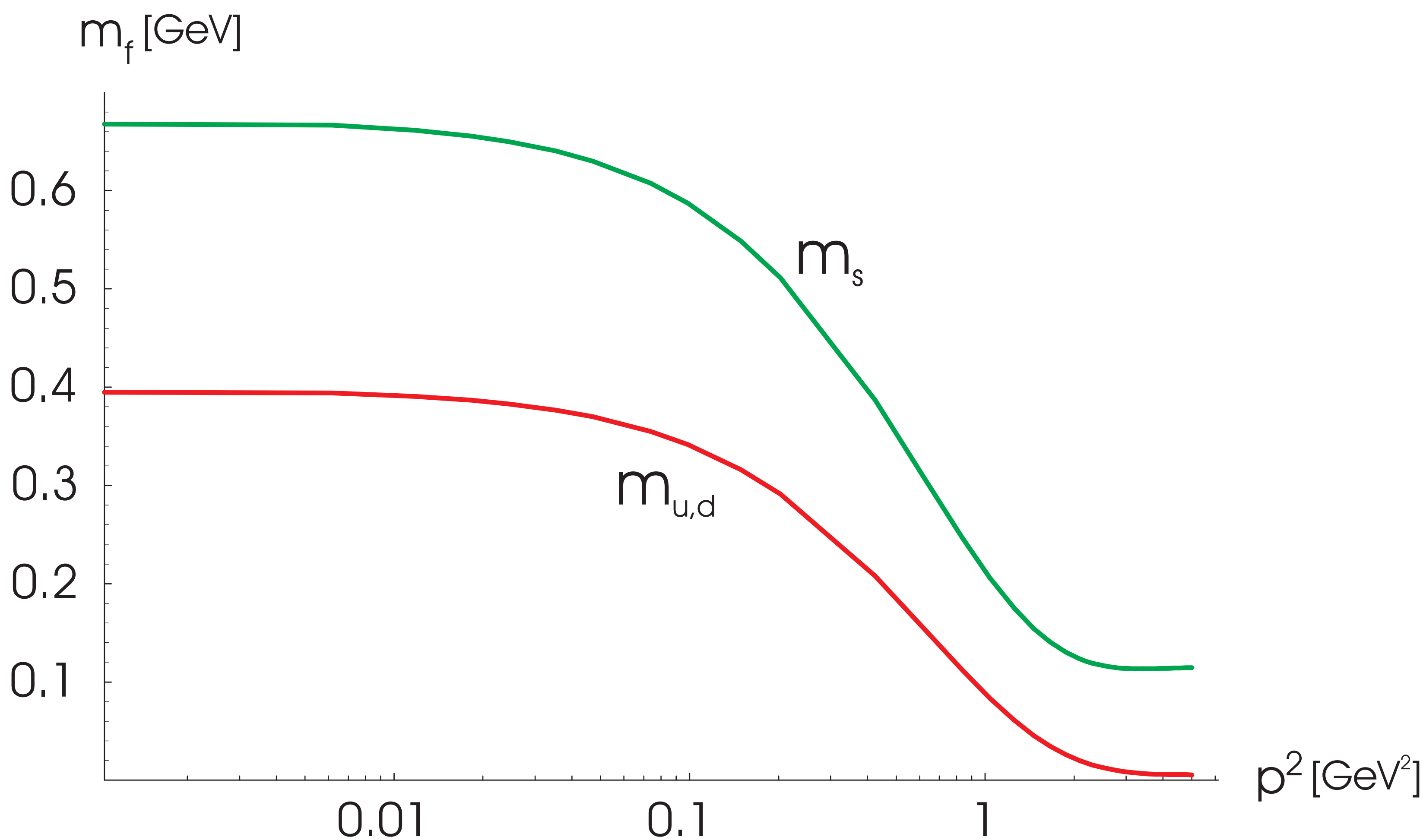
Nonperturbative dynamical propagator dressing

- $\longrightarrow$ Dynamical Chiral Symmetry Breaking (DChSB)



DChSB = nonperturb. generation of large quark masses ...

- ... even in the chiral limit ($\widetilde{m}_f \to 0$), where the octet pseudoscalar mesons are Goldstone bosons of DChSB!



Separable model: "non-anomalous" results (at $T = 0$)

- Model parameter values reproducing experimental data:
- $\widetilde{m}_{u,d} = 5.5$ MeV, $\Lambda_0 = 758$ MeV, $\Lambda_1 = 961$ MeV, $p_0 = 600$ MeV, $D_0\Lambda_0^2 = 219$, $D_1\Lambda_1^4 = 40$ (fixed by fitting M_π, f_π, M_ρ, $g_{\rho\pi^+\pi^-}$, $g_{\rho e^+e^-}$ $\to$ pertinent predictions $a_{u,d} = 0.672$, $b_{u,d} = 660$ MeV, i.e., $m_{u,d}(p^2)$, $\langle\bar{u}u\rangle$)
- $\widetilde{m}_s = 115$ MeV (fixed by fitting $M_K \to$ predictions $a_s = 0.657$, $b_s = 998$ MeV, i.e., $m_s(p^2)$, $\langle\bar{s}s\rangle$, $M_{s\bar{s}}$, f_K, $f_{s\bar{s}}$)
- Summary of results (all in GeV) for $q = u, d, s$ and pseudoscalar mesons without the influence of gluon anomaly:

PS	M_{PS}	M_{PS}^{exp}	f_{PS}	f_{PS}^{exp}	$m_q(0)$	$-\langle q\bar{q}\rangle_0^{1/3}$
π	0.140	0.1396	0.092	0.0924 ± 0.0003	0.398	0.217
K	0.495	0.4937	0.110	0.1130 ± 0.0010		
$s\bar{s}$	0.685		0.119		0.672	

Anomaly and mixing in $\eta - \eta'$ complex

- present approach yields mass^2 eigenvalues
  $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_{K}, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$
- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_{\pi}$. I is a good quantum number!
- $\longrightarrow$ recouple into the familiar $I_3 = 0$ octet-singlet basis

$$
\begin{aligned}
|\pi^0\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \ , \\
|\eta_8\rangle &= \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \ , \\
|\eta_0\rangle &= \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \ .
\end{aligned}
$$

Anomaly and mixing in $\eta - \eta'$ complex

- the "non-anomalous" (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η's is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M^2_\pi & 0 & 0 \\ 0 & M^2_{88} & M^2_{80} \\ 0 & M^2_{08} & M^2_{00} \end{pmatrix}$$

$$M^2_{88} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle \equiv M^2_{\eta_8} = \frac{2}{3}(M^2_{s\bar{s}} + \frac{1}{2} M^2_\pi),$$

$$M^2_{80} \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M^2_{08} = \frac{\sqrt{2}}{3}(M^2_\pi - M^2_{s\bar{s}})$$

$$M^2_{00} \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2} M^2_{s\bar{s}} + M^2_\pi),$$

- in order to avoid the U$_A$(1) problem, U$_A$(1) symmetry must ultimately be broken by gluon anomaly at least at the level of the masses

Anomaly and mixing in $\eta - \eta'$ complex

- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly!
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\rightarrow$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \quad \text{does not vanish in the chiral limit.}$$

- 3β, the anomalous mass of η_0, is related to the topological susceptibility of the vacuum. It is fixed by phenomenology or taken from the lattice calculations.

Anomaly and mixing in $\eta - \eta'$ complex

- we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$. $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.
- Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- $\rightarrow$ In the isospin limit, one can always restrict to 2×2 submatrix of etas

Anomaly and mixing in $\eta - \eta'$ complex

- nonstrange (*NS*) – strange (*S*) basis

$$|\eta_{NS}\rangle = \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \ ,$$
$$|\eta_S\rangle = |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \ .$$

- the η–η' matrix in this basis is

$$\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S \eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{pmatrix}$$

- *NS*–*S* mixing relations

$$|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \ , \qquad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \ .$$

$$\theta = \phi - \arctan\sqrt{2}$$

Anomaly and mixing in $\eta - \eta'$ complex

- Let lowercase m_M's denote the empirical mass of meson M. From our calculated, model mass matrix in *NS*–*S* basis, we form its empirical counterpart $\hat{m}^2_{\rm exp}$ by
- *i)* obvious substitutions $M_{u\bar{u}} \equiv M_\pi \to m_\pi$, $M_{s\bar{s}} \to m_{s\bar{s}}$
- *ii)* by noting that $m_{s\bar{s}}$, the "empirical" mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m^2_{s\bar{s}} \approx 2m^2_K - m^2_\pi$ due to GMOR. Then,

$$\hat{m}^2_{\rm exp} = \left[\begin{array}{cc} m^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m^2_K - m^2_\pi + \beta X^2 \end{array} \right] \xrightarrow[\phi_{\rm exp}]{} \left[\begin{array}{cc} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{array} \right].$$

Anomaly and mixing in $\eta - \eta'$ complex

- requiring that the experimental trace $(m_\eta^2 + m_{\eta'}^2)_{exp} \approx 1.22$ GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\rm fit} = \frac{1}{2+X^2}[(m_\eta^2 + m_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)]$
- But better get β from lattice! Then no free parameters!
- the trace of the empirical $\hat{m}^2_{\rm exp}$ demands the 1^{st} equality in

$$\beta(2+X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2}\chi \quad (2^{\rm nd}\text{equality} = \text{WV relation})$$

- we can then determine the mixing angle ϕ through

$$\tan 2\phi = \frac{2\, M^2_{\eta_S \eta_{NS}}}{M^2_{\eta_S} - M^2_{\eta_{NS}}} = \frac{2\sqrt{2}\beta X}{M^2_{\eta_S} - M^2_{\eta_{NS}}},$$

$$M^2_{\eta_{NS}} = M^2_{u\bar{u}} + 2\beta = M^2_\pi + 2\beta, \quad M^2_{\eta_S} = M^2_{s\bar{s}} + \beta X^2 = M^2_{s\bar{s}} + \beta\frac{f_\pi^2}{f_{s\bar{s}}^2}$$

Anomaly and mixing in $\eta - \eta'$ complex

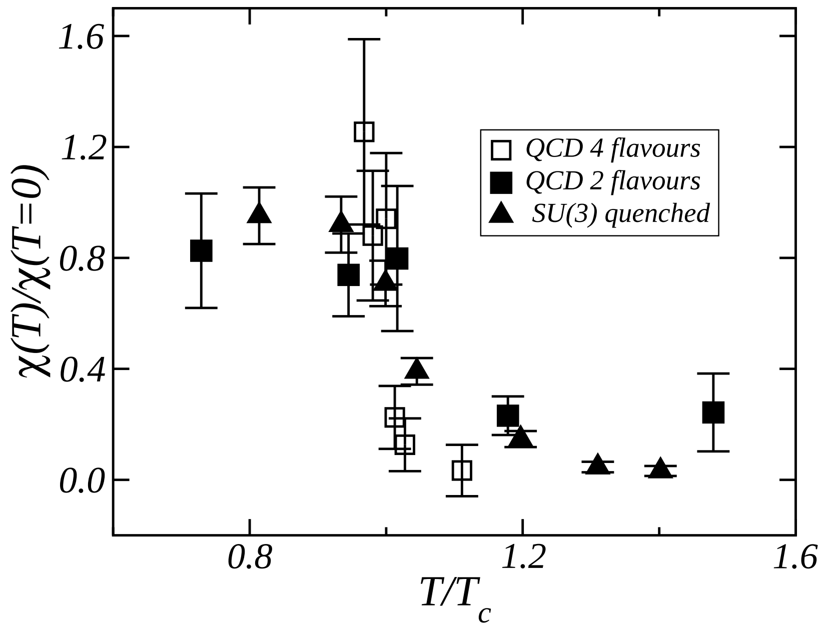
- The diagonalization of the $NS - S$ mass matrix then finally gives us the *calculated* η and η' masses:

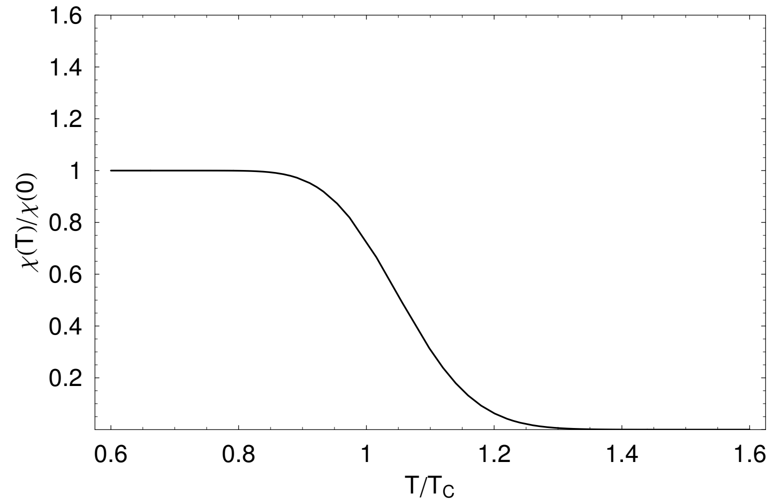
$$
\begin{aligned}
M_\eta^2 &= \cos^2\phi \, M_{\eta_{NS}}^2 - \sqrt{2}\beta X \sin 2\phi + \sin^2\phi \, M_{\eta_S}^2 \\
M_{\eta'}^2 &= \sin^2\phi \, M_{\eta_{NS}}^2 + \sqrt{2}\beta X \sin 2\phi + \cos^2\phi \, M_{\eta_S}^2
\end{aligned}
$$

- Equivalently, from the secular determinant,

$$
\begin{aligned}
M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \\
M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right]
\end{aligned}
$$

Topological susceptibility of QCD vacuum





$$\chi = \int d^4x \, \langle q(x) q(0) \rangle \, , \qquad q(x) = \frac{g^2}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x)$$

- $q(x)$ = topological charge density operator

Separable model results on η and η' mesons (at $T = 0$)

	$\beta_{\rm fit}$	$\beta_{\rm latt.}$	Exp.
θ	-12.22°	-13.92°	
M_η	548.9	543.1	547.75
$M_{\eta'}$	958.5	932.5	957.78
X	0.772	0.772	
3β	0.845	0.781	

- masses are in units of MeV, 3β in units of GeV^2 and the mixing angles are dimensionless.
- $\beta_{\rm latt.}$ was obtained from $\chi(T=0) = (175.7\ \rm MeV)^4$
- $X = f_\pi / f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.

Shore's generalization of WV – valid to all orders in $1/N_c$

- Inclusion of gluon anomaly in DGMOR relations $\rightarrow$

$$(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 = \tfrac{1}{3}\big(f_\pi^2 m_\pi^2 + 2 f_K^2 m_K^2\big) + 6A \qquad (1)$$

$$f^{0\eta'} f^{8\eta'} m_{\eta'}^2 + f^{0\eta} f^{8\eta} m_\eta^2 = \tfrac{2\sqrt{2}}{3}\big(f_\pi^2 m_\pi^2 - f_K^2 m_K^2\big) \qquad (2)$$

$$(f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_\eta^2 = -\tfrac{1}{3}\big(f_\pi^2 m_\pi^2 - 4 f_K^2 m_K^2\big) \qquad (3)$$

- $A = \chi + \mathcal{O}(\frac{1}{N_c})$ = full QCD topological charge. (1)+(3)$\rightarrow$

$$(f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 + (f^{8\eta})^2 m_\eta^2 + (f^{8\eta'})^2 m_{\eta'}^2 - 2 f_K^2 m_K^2 = 6A$$

- Then, large N_c limit and $f^{0\eta}, f^{8\eta'} \rightarrow 0$ as well as $f^{0\eta'}, f^{8\eta}, f_K \rightarrow f_\pi$ recovers the standard WV.

η' and η have 4 independent decay constants

$$f^0_{\eta'}, f^8_{\eta}, f^0_{\eta}, f^8_{\eta'}: \qquad \langle 0|A^{a\,\mu}(x)|P(p)\rangle = i f^a_P\, p^\mu e^{-ip\cdot x}, \quad a=8,0; \;\; P=\eta,\eta' .$$

- Equivalently, one has 4 related but different constants $f^{NS}_{\eta'}, f^{NS}_{\eta}, f^{S}_{\eta}, f^{S}_{\eta'}$ if instead of octet and singlet axial currents ($a=8,0$) one takes this matrix element of the nonstrange-strange axial currents ($a=NS,S$)

$$A^\mu_{NS}(x) = \frac{1}{\sqrt{3}} A^{8\,\mu}(x) + \sqrt{\frac{2}{3}} A^{0\,\mu}(x) = \frac{1}{2}\left(\bar{u}(x)\gamma^\mu\gamma_5 u(x) + \bar{d}(x)\gamma^\mu\gamma_5 d(x)\right) ,$$

$$A^\mu_{S}(x) = -\sqrt{\frac{2}{3}} A^{8\,\mu}(x) + \frac{1}{\sqrt{3}} A^{0\,\mu}(x) = \frac{1}{\sqrt{2}} \bar{s}(x)\gamma^\mu\gamma_5 s(x) ,$$

$$\begin{bmatrix} f^{NS}_{\eta} & f^{S}_{\eta} \\ f^{NS}_{\eta'} & f^{S}_{\eta'} \end{bmatrix} = \begin{bmatrix} f^{8}_{\eta} & f^{0}_{\eta} \\ f^{8}_{\eta'} & f^{0}_{\eta'} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} ,$$

$$a, P = NS, S: \qquad \langle 0|A^\mu_{NS}(x)|\eta_{NS}(p)\rangle = i f_{NS}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{NS}(x)|\eta_{S}(p)\rangle = 0 ,$$

$$a, P = NS, S: \qquad \langle 0|A^\mu_{S}(x)|\eta_{S}(p)\rangle = i f_{S}\, p^\mu e^{-ip\cdot x} , \qquad \langle 0|A^\mu_{S}(x)|\eta_{NS}(p)\rangle = 0 ,$$

- Note: in our approach, $f_{NS} = f_{u\bar{u}} = f_{s\bar{s}} = f_\pi$, $f_S = f_{s\bar{s}}$ are calculated quantities

Two Mixing Angles and FKS one-angle scheme

- Any 4 η-η' decay constants conveniently parametrized in terms of two decay constants and two angles:

$$f_\eta^8 = \cos\theta_8\, f_8 \,, \qquad f_\eta^0 = -\sin\theta_0\, f_0 \,, \qquad f_\eta^{NS} = \cos\phi_{NS}\, f_{NS} \,, \qquad f_\eta^S = -\sin\phi_S\, f_S \,,$$

$$f_{\eta'}^8 = \sin\theta_8\, f_8 \,, \qquad f_{\eta'}^0 = \cos\theta_0\, f_0 \,, \qquad f_{\eta'}^{NS} = \sin\phi_{NS} f_{NS} \,, \qquad f_{\eta'}^S = \cos\phi_S f_S$$

- Big practical difference between 0-8 and NS-S schemes:

- while θ_8 and θ_0 differ a lot from each other and from $\theta \approx (\theta_8 + \theta_0)/2$, FKS showed that $\phi_{NS} \approx \phi_S \approx \phi$.

$$\begin{bmatrix} f_\eta^{NS} & f_\eta^S \\ f_{\eta'}^{NS} & f_{\eta'}^S \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} .$$

For four decay constants, can use FKS one-angle scheme!

- we can relate $\{f_\eta^8, f_{\eta'}^8, f_\eta^0, f_{\eta'}^0\}$ with $\{f_{NS}, f_S\} = \{f_\pi, f_{s\bar{s}}\}$:

$$\begin{bmatrix} f_\eta^8 & f_\eta^0 \\ f_{\eta'}^8 & f_{\eta'}^0 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} f_{NS} & 0 \\ 0 & f_S \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

- Some other useful relations between quantities of NS-S (FKS) and 0-8 schemes:

$$f_8 = \sqrt{\frac{1}{3} f_{NS}^2 + \frac{2}{3} f_S^2}\,, \qquad \theta_8 = \phi - \arctan\left(\frac{\sqrt{2} f_S}{f_{NS}}\right)\,,$$

$$f_0 = \sqrt{\frac{2}{3} f_{NS}^2 + \frac{1}{3} f_S^2}\,, \qquad \theta_0 = \phi - \arctan\left(\frac{\sqrt{2} f_{NS}}{f_S}\right)\,.$$

- $\rightarrow$ We can solve 3 Shore's equations for ϕ, m_η and $m_{\eta'}$.

Separable model results of Shore's generalization

- The non-anomalous results are the same as before

$\chi(0)$	-(190 MeV)^4	-(182 MeV)^4	-(175.7 MeV)^4	[EXP]	**with WV**
m_η [MeV]	525.2	516.1	506.9	547.7	543.1
$m_{\eta'}$ [MeV]	975.0	913.1	868.7	957.8	932.5
ϕ	46.89°	43.80°	40.86°		40.82°

$\chi(0)$	-(190 MeV)^4	-(182 MeV)^4	-(175.7 MeV)^4	Shore	**with WV**
θ_0	-0.66°	-3.76°	-6.69°	-12.3°	-6.73°
θ_8	-14.45°	-17.54°	-20.47°	-20.1°	-20.52°
f_0 [MeV]	101.8	101.8	101.8	106.6	101.8
f_8 [MeV]	110.7	110.7	110.7	104.8	110.7
f_0^η [MeV]	1.2	6.7	11.9	22.8	11.9
$f_0^{\eta'}$ [MeV]	101.8	101.6	101.1	104.2	101.1
f_8^η [MeV]	107.2	105.6	103.7	98.4	103.7
$f_8^{\eta'}$ [MeV]	-27.6	-33.4	-38.7	-36.1	-38.8

Jain-Munczek model: old results vs. Shore's generalization

	A	B	C	D	E
X	1.0	0.673	0.805		
3β	0.707	0.865	0.801		
θ	-19.5°	-11.1°	-14.9°	-12.8°	-
m_η	0.5048	0.5777	exp.		0.54730
$m_{\eta'}$	0.9809	0.9398	exp.		0.95778
$\Gamma(\eta \to \gamma\gamma)$	0.63	0.44	0.52	0.48	0.46 ± 0.04
$\Gamma(\eta' \to \gamma\gamma)$	3.61	4.61	4.16	4.41	4.26 ± 0.19

$\chi(0)$	-(190 MeV)^4	-(175.7 MeV)^4	[EXP]
m_η [MeV]	497.6	484.4	547.7
$m_{\eta'}$ [MeV]	922.5	814.8	957.8
ϕ	51.80°	46.25°	
θ	-2.94°	-8.49°	

$G^{\text{eff}}_{\mu\nu}$ from $\langle A^2 \rangle$: results from WV vs. Shore's generalization

$\chi(0)$	-(190 MeV)^4	-(175.7 MeV)^4	[EXP]	**with WV**
m_η [MeV]	497.8	484.4	547.7	577.1
$m_{\eta'}$ [MeV]	926.7	818.9	957.8	932.0
ϕ	51.38°	45.83°		39.56°
θ	-3.36°	-8.91°		-15.18°

$\chi(0)$	-(190 MeV)^4	-(175.7 MeV)^4	Shore
θ_P	-2.80°	-8.36°	-16.5°
θ_0	6.70°	1.15°	-12.3°
θ_8	-12.31°	-17.86°	-20.1°
f_0 [MeV]	108.0	108.0	106.6
f_8 [MeV]	121.1	121.1	104.8
f_0^η [MeV]	-12.6	-2.17	22.8
$f_0^{\eta'}$ [MeV]	107.3	108.0	104.2
f_8^η [MeV]	118.4	115.3	98.4
$f_8^{\eta'}$ [MeV]	-25.8	-37.2	-36.1

Summary

- Sketched Dyson-Schwinger approach to quark-hadron physics & convenient concrete dynamical models
- Results for dressed quarks, pions and kaons at $T = 0$
- Anomaly and mixing in the $\eta - \eta'$ complex
- Results on $\eta - \eta'$ complex (at $T = 0$) via Witten-Veneziano relation
- Shore's generalization in principle valid to all orders in $\frac{1}{N_c}$
- Applying Shore's scheme – in practice, with approximations: $A \to \chi$, FKS one-angle scheme
- In the present $T = 0$ case, standard Witten-Veneziano relation gives better results, probably for reasons of consistency
- True advantages of Shore's generalization – at $T > 0$.