Counterterm resummation for 2PI-approximation in constant background

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Plan of the talk:

- Resummation and renormalisation:
  towards functional methods competitive with lattice simulations
- Counterterm construction for 2-loop 2PI approximation
  to the effective action of real scalar field in the broken symmetry phase
- Adding the basket-ball diagram: complete $\mathcal{O}(\lambda^2)$ skeleton analysis
- Summary of present stage and survey of subjects in progress

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Progress in RR (renormalizable resummations)

Where can functional approaches be competitive with lattice methods?

- Thermodynamics of electroweak and strong matter
  at large baryonic, isotopic and strange densities

- Real time, out of equilibrium quantum processes
  e.g. inflation, reheating,
  birth and death of extended galactic objects, like cosmic strings

Problems of resummed perturbative series

- Realisation of symmetries (e.g. Ward-Takahashi identities, gapless approximation in broken phase, gauge fixing parameter dependence of 2PI applications)

- Simple practical implementation of the renormalisation program

Two approaches to renormalised 2PI equations

Generic SD-equation for the propagator

$$iG^{-1}(p) = p^2 - m^2 - \kappa v^2 - \delta m^2 - \delta\kappa v^2 - 2\frac{\delta\Gamma_2}{\delta G(p)} \equiv p^2 - m^2 - \kappa v^2 - \Sigma_{ren}$$

Approach I

By appropriate number of (divergence) substractions find $\frac{\delta\Gamma_2}{\delta G(p)}_{ren}$. Solve

$$2\frac{\delta\Gamma_2}{\delta G(p)}_{ren}[\Sigma_{ren}] = \Sigma_{ren}$$

Approach II

Find explicit regularized expressions for the counterterms (e.g. $\delta m^2, \delta\kappa$) and also for all contributions to the regularized $\frac{\delta\Gamma_2}{\delta G(p)}_{reg}$. Solve

$$\delta m^2 + \delta\kappa v^2 + 2\frac{\delta\Gamma_2}{\delta G(p)}_{reg}[\delta m^2, \delta\kappa, \Sigma_{ren}] = \Sigma_{ren}$$

Counterterm construction for 2-loop 2PI approximation to the effective potential of real scalar model

$$V[v,G] = \frac{1}{2}m^2v^2 + \frac{\lambda}{24}v^4 - \frac{i}{2}\int_p \left[\ln G^{-1}(p) + D^{-1}(p)G(p)\right]$$
$$+\frac{\lambda}{8}\left(\int_p G(p)\right)^2 - \frac{i\lambda^2}{12}v^2\int_k\int_p G(p)G(k)G(p-k) + V_{ct}[v,G],$$

$$iD^{-1}(p) = p^2 - m^2 - \lambda v^2/2$$

Counterterm functional given by

$$V_{ct}[v,G] = \frac{1}{2}\delta m_0^2 v^2 + \frac{\delta\lambda_4}{24}v^4 + \frac{1}{2}\left(\delta m_2^2 + \frac{\delta\lambda_2}{2}v^2\right)\int_p G(p) + \frac{\delta\lambda_0}{8}\left(\int_p G(p)\right)^2.$$

Comments:

- No new counterterm relative to Hartree approximation is needed.
- Independent counterterms are introduced for each 2PI-piece.

Propagator equation

$$iG^{-1}(p) = p^2 - m^2 - \delta m_2^2 - \frac{1}{2}(\lambda + \delta\lambda_2)v^2 - \frac{1}{2}(\lambda + \delta\lambda_0)T[G] - \frac{1}{2}\lambda^2 v^2 I(p, G).$$

Tadpole ($T[G]$) and bubble ($I[p, G]$) contributions split into divergent and finite pieces:

$$T[G] := \int_k G(k) = T_{\text{div}} + T_F[G],$$

$$I(p, G) := -i \int_k G(k)G(k+p) = I_{\text{div}} + I_F(p, G)$$

Parametrisation of the exact propagator:

$$iG^{-1}(p) = p^2 - M^2 - \Pi(p)$$

Self-energy split into p-independent (M^2) and p-dependent ($\Pi(p)$) pieces

$$M^2 = m^2 + \frac{\lambda}{2}v^2 + \frac{\lambda}{2}T_F[G], \qquad \Pi(p) = \frac{\lambda^2}{2}v^2 I_F(p, G)$$

$\lim_{p\to\infty} \Pi(p) \sim [\ln(p^2)]^\alpha$: Unchanged large-p asymptotics of $G^{-1}(p)$

Equation of state and renormalisation conditions

Equation of state

$$v\left(m^2+\delta m_0^2+\frac{1}{6}(\lambda+\delta\lambda_4)v^2+\frac{1}{2}(\lambda+\delta\lambda_2)T[G]+\frac{1}{6}\lambda^2 S(0,G)\right)=0.$$

with

$$S(p,G):=-i\int_k\int_q G(k)G(q)G(k+q+p)=S_{\text{div}}(p,G)+S_F(p,G).$$

Renormalisation conditions:

$$0=\delta m_2^2+\frac{1}{2}\delta\lambda_2 v^2+\frac{1}{2}(\lambda+\delta\lambda_0)T_{\text{div}}+\frac{1}{2}\delta\lambda_0 T_F[G]+\frac{1}{2}\lambda^2 v^2 I_{\text{div}},$$

$$0=\delta m_0^2+\frac{1}{6}\delta\lambda_4 v^2+\frac{1}{2}(\lambda+\delta\lambda_2)T_{\text{div}}+\frac{1}{2}\delta\lambda_2 T_F[G]+\frac{1}{6}\lambda^2 S_{\text{div}}(0,G).$$

Coefficients of v^0 and v^2 correspond in this scheme to overall divergences

Vanishing of the divergent coefficient of $T_F[G]$ should also be imposed
Independent criterium which cancels a subdivergence

$\rightarrow$ 1+1=2 extra conditions

Divergence analysis I.

Auxiliary propagator with the same large-p asymptotics as $G^{-1}(p)$ has introduces also normalisation scale M_0^2:

$$iG^{-1}_{aux}(p) = p^2 - M_0^2$$

Divergent piece of each integral should be expressed through G_{aux}:

Bubble integral $I(p,G)$:

$$I_{\text{div}} = -i\int_k G^2_{aux}(k)\Big|_{\text{div}} =: T_d^{(0)}, \qquad I_F(p) = -i\int_k \left[G(k)G(k+p) - G^2_{aux}(k)\right]$$

Tadpole and setting sun integrals analysed with help of an appropriate representation for the propagator:

$$G(p) = G_{aux}(p) + \delta G(p),$$

$$\delta G(p) = -iG^2_{aux}(p)\left(M^2 - M_0^2 + \frac{\lambda^2 v^2}{2} I_{aux,F}(p)\right) + G_r(p), \qquad G_r(p) \sim p^{-6}$$

Divergent piece of the tadpole integral $T[G]$:

$$T[G] = \int_k G_{aux}(k) + (M^2 - M_0^2)T_d^{(0)} + \frac{1}{2}\lambda^2 v^2 (-i)\int_k G^2_{aux}(k) I_{aux F}(k) + T_F^{(1)}[G]$$

Divergence analysis II.

$$T_{\text{div}} = T_d^{(2)} + (M^2 - M_0^2)T_d^{(0)} + \tfrac{1}{2}\lambda^2 v^2 T_d^{(I)}$$

$$T_d^{(2)} := T[G_{aux}] = \int_p G_{aux}(p)\Big|_{\text{div}}, \qquad T_d^{(I)} := -i\int_p G_{aux}^2(p) I_{aux,F}(p)\Big|_{\text{div}}$$

Setting-sun integral $S(p=0,G) = -i\int_k\int_q G(k)G(q)G(k+q)$

Rewritten with the replacement
$G(p) = G_{aux}(p) + \delta G(p)$ on each of the three propagators:

$$S(p=0,G) = S_{aux}(0) + 3\int_k \delta G(k)(I_{div} + I_{aux,F}(k)) + S_F^{(1)}.$$

$$S_{\text{div}}(0,G) = S_{aux}(0) + 3(T[G] - T_d^{(2)})T_d^{(0)} + 3(M^2 - M_0^2)T_d^{(I)} + \frac{3}{2}\lambda^2 v^2 T_d^{(I,2)}$$

$$T_d^{(I,2)} := -i\int_k G_{aux}^2(k) I_{aux,F}^2(k)\Big|_{\text{div}},$$

Algebraic equations for the non-perturbative counterterms

4 equations ensuring cancellation of overall divergencies determine $\delta m_0^2, \delta m_2^2, \delta\lambda_2, \delta\lambda_4$ with input: $\delta\lambda_0$

$$0 = \delta m_2^2 + \frac{1}{2}(\lambda + \delta\lambda_0)\left[T_d^{(2)} + (m^2 - M_0^2)\right] T_d^{(0)},$$

$$0 = \delta\lambda_2 + \frac{1}{2}\lambda(\lambda + \delta\lambda_0)\left(T_d^{(0)} + \lambda T_d^{(I)}\right) + \lambda^2 T_d^{(0)},$$

$$0 = \delta m_0^2 + \frac{1}{2}(\lambda + \delta\lambda_2)\left[T_d^{(2)} + (m^2 - M_0^2) T_d^{(0)}\right] + \frac{1}{2}\lambda^2 (m^2 - M_0^2)\left((T_d^{(0)})^2 + T_d^{(I)}\right) + \frac{1}{6}\lambda^2 S_{aux}(0),$$

$$0 = \delta\lambda_4 + \frac{3}{2}\lambda(\lambda + \delta\lambda_2 + \lambda^2 T_d^{(0)})\left(T_d^{(0)} + \lambda T_d^{(I)}\right) + \frac{3}{2}\lambda^3\left(T_d^{(I)} + \lambda T_d^{(I,2)}\right)$$

2 equations for cancelling divergent coefficients of T_F determine $\delta\lambda_0$

$$0 = \delta\lambda_0 + \frac{1}{2}\lambda(\lambda + \delta\lambda_0) T_d^{(0)},$$

and give a relation for $\delta\lambda_2$:

$$0 = \delta\lambda_2 + \frac{1}{2}\lambda(\lambda + \delta\lambda_2) T_d^{(0)} + \frac{1}{2}\lambda^3\left((T_d^{(0)})^2 + T_d^{(I)}\right) + \lambda^2 T_d^{(0)}$$

Check: they are mutually consistent

Adding the basket ball: complete $\mathcal{O}(\lambda^2)$ skeleton truncation

$$iG^{-1}(p) = p^2 - \Sigma(p), \qquad \Sigma(p) = M^2 + \Pi_0(p) + \Pi_2(p)$$
$$M^2 = m^2 + \frac{1}{2}\lambda v^2 + \frac{1}{2}\lambda T_F[G], \qquad \Pi_0(p) = \frac{1}{2}\lambda^2 v^2 I_F(p)$$
$$\Pi_2(p) = \frac{1}{6}\lambda^2 S_F(p), \qquad \Pi_2 = \Pi_a(p) + \Pi_{20} + \Pi_r,$$

$$\Pi_a \approx p^2(\ln p^2)^{c_1}, \quad \Pi_{20} \approx (\ln p^2)^{c_2}, \quad \Pi_r \approx p^{-2}$$

Divergence cancellation in the propagator

$$0 = \delta m_2^2 + \tfrac{1}{2}\delta\lambda_2 v^2 + \tfrac{1}{2}(\lambda + \delta\lambda_0)T_{\text{div}} + \tfrac{1}{2}\delta\lambda_0 T_F[G] + \tfrac{1}{2}\lambda^2 v^2 I_{\text{div}} + \tfrac{1}{6}\lambda^2 S_{\text{div}}(p) - p^2\delta Z$$

Divergence analysis with the auxiliary propagator $iG_{aux}^{-1} = p^2 - M_0^2 - \Pi_a(p)$

$$T[G]_{\text{div}} = T_a^{(2)} + (M^2 - M_0^2)T_a^{(0)} + \frac{1}{2}\lambda^2 v^2 T_a^{(I)} - i\int_k G_a^2(k)\Pi_{2,0}(k)\Big|_{\text{div}}$$

$$S_{\text{div}}(p) - \frac{6}{\lambda^2}p^2\delta Z = S_{a,\text{div}}(0) + 3T_a^{(0)}(T[G] - T_a^{(2)}) + 3\big(M^2 - M_0^2\big)T_a^{(I)}$$
$$+\frac{3}{2}\lambda^2 v^2 T_a^{(I,2)} - 3i\int_k G_a^2(k) I_{a,F}(k)\Pi_{2,0}(k)\Big|_{\text{div}}$$

Equation of $\Pi_{20}(p)$

$$\Pi_{20}(p) = -\frac{i}{2}\int_k G_a^2(k)K(p,k)\left[M^2 - M_0^2 + \frac{\lambda^2}{2}v^2 I_{a,F}(k) + \Pi_{20}(k)\right] \text{ with}$$

$$K(p,k) = \frac{\lambda^2}{2}\Big[I_{a,F}(k+p) + I_{a,F}(k-p) - 2I_{a,F}(k)\Big]$$

Linear expression in terms of $M^2 - M_0^2$ and $\frac{1}{4}\lambda^2 v^2$:

$$\Pi_{20}(p) = \frac{1}{2}(M^2 - M_0^2)\Gamma(p) + \frac{1}{4}\lambda^2 v^2 \tilde{\Gamma}(p)$$

where

$$\Gamma(p) = -i\int_k \Gamma(p,k)G_a^2(k), \quad \tilde{\Gamma}(p) = \int_k \Gamma(p,k)G_a^2(k)I_{a,F}(k).$$

Bethe-Salpeter equation for $\Gamma(p,k)$:

$$\Gamma(p,k) = K(p,k) - \frac{i}{2}\int_q G^2(q)K(p,q)\Gamma(q,k).$$

Introduce moments of the asymptotic propagator:

$$D_0 = -i\int_k G_a^2(k)\Gamma(k)\Big|_{\text{div}}, \quad \tilde{D}_0 = -i\int_k G_a^2(k)\tilde{\Gamma}(k)\Big|_{\text{div}}$$

$$D_1 = -i\int_k G_a^2(k)I_{aF}(k)\Gamma_0(k)\Big|_{\text{div}}, \quad \tilde{D}_1 = -i\int_k G_a^2(k)I_{aF}(k)\tilde{\Gamma}(k)\Big|_{\text{div}}$$

Detailed counterterm equations for the propagator

$$0 = \delta m_2^2 + \tfrac{1}{2}(\lambda + \delta\lambda_0)\left[T_a^{(2)} + (m^2 - M_0^2)T_a^{(0)}\right] + \frac{\lambda^2}{6}S_{a,\mathsf{div}}(0)$$
$$+(m^2 - M_0^2)\left\{\frac{\lambda^2}{2}\left[(T_a^{(0)})^2 + T_a^{(I)} + \tfrac{1}{2}\tilde{D}_1\right] + \tfrac{1}{4}\big(\lambda + \delta\lambda_0 + \lambda^2 T_a^{(0)}\big)D_0\right\},$$

$$0 = \delta\lambda_2 + \tfrac{1}{2}\lambda(\lambda + \delta\lambda_0)\big(T_a^{(0)} + \lambda T_a^{(I)}\big) + \lambda^2 T_a^{(0)}$$
$$+\tfrac{1}{2}\lambda^3\Big[(T_a^{(0)})^2 + T_a^{(I)} + \lambda\left(T_a^{(0)}T_a^{(I)} + T_a^{(I,2)}\right)\Big]$$
$$+\tfrac{1}{4}\lambda\Big[\left(\lambda + \delta\lambda_0 + \lambda^2 T_a^{(0)}\right)(D_0 + \lambda D_1) + \lambda^2(\tilde{D}_1 + \lambda D_2)\Big]$$

Coefficent of T_F:

$$0 = \delta\lambda_0 + \tfrac{1}{2}\lambda(\lambda + \delta\lambda_0)T_a^{(0)} + \lambda^2\left[T_a^{(0)} + \tfrac{1}{2}\lambda\left((T_a^{(0)})^2 + T_a^{(I)}\right)\right]$$
$$+\tfrac{1}{4}\lambda\Big[\left(\lambda + \delta\lambda_0 + \lambda^2 T_a^{(0)}\right)D_0 + \lambda^2\tilde{D}_1\Big].$$

Comparing with the the coefficient of T_F in the equation of state:

$$\delta\lambda_0 = \delta\lambda_2$$

Consistency ?

$$\delta\lambda_0 - \delta\lambda_2 = \frac{\lambda^3}{2}\left[1 + \frac{\lambda}{2}\left(T_a^{(0)} + \frac{1}{2}D_0\right)\right]^{-1}\left\{T_a^{(I)} + \frac{1}{2}D_1 + \lambda\left(T_a^{(I,2)} + \frac{1}{2}D_2\right)\right.$$

$$\left. + \frac{1}{2}\lambda^2\left[\left(T_a^{(I,2)} + \frac{1}{2}D_2\right)\left(T_a^{(0)} + \frac{1}{2}D_0\right) - \left(T_a^{(I)} + \frac{1}{2}D_1\right)\left(T_a^{(I)} + \frac{1}{2}\tilde{D}_1\right)\right]\right\}$$

=? 0

Remark:

The counterterms of the propagator $\delta\lambda_0, \delta\lambda_2$ can be analysed also with the method of iterative renormalisation (Blaizot, Iancu, Reinosa, 2004) where one looks for the self-energy in form of infinite series:

$$\delta\lambda_0 = \sum_n \delta\lambda_0^{(n)}, \qquad \delta\lambda_2 = \sum_n \delta\lambda_2^{(n)}$$

and solves the gap equations iteratively. It yields

$$\delta\lambda_0 = \delta\lambda_2$$

Indirect argument for the validity of the consistency relation!

Other two counterterms of the equation of state:

$$\delta m_2^2 = \delta m_0^2, \qquad \delta\lambda_4 = 3(\delta\lambda_2 + \lambda^2 T_a^{(0)}).$$

CONCLUSIONS

- Simple and explicit construction of countertems
- Consistency checks on the non-perturbative renormalisation
- Generalisation to $O(N)$ model (in principle to any global symmetry)

OUTLOOK

Application to models of phenomenological interest

Addressing further conceptual questions:

- construction of gapless approximation beyond Hartree truncation
- renormalisation of NLO large N approximation
- counterterm construction for nPI approximation