# On ubiquitous long living breather type lumps


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# Prolegomena

- Spatially localized, time periodic solutions of various (scalar, vector, . . .) nonlinear wave equations are of great mathematical and physical interest, from nonlinear optics to field theories and in general relativity
- A great deal of rigorous results is known about the existence of localized, time periodic solutions of various nonlinear wave equations (standing waves, etc.) for compact domains.
- for non-compact domains much less results are available on the (non)existence of such solutions
- one expects that generically no such (non-radiative) solutions exist
  physicist's naive (?) argument: if a system can radiate it does radiate

# Prolegomena

- It has been shown that neither the Yang-Mills equations nor the vacuum Einstein eqs. admit non-radiative solutions – $\nexists$ glueballs (Coleman, '77), (Gibbons, Stuart '84)
- SG breathers: a whole family of localized, non radiative solutions is known in the sine-Gordon theory in 1 spatial dimension:
$$-\frac{\partial^2\phi}{\partial t^2}+\frac{\partial^2\phi}{\partial x^2}=\sin(\phi)$$
and a typical family of breathers can be written as:
$$\phi(x,t)=4\arctan\left(\frac{\epsilon\sin\omega t}{\omega\cosh\epsilon x}\right)\,,\quad \epsilon^2+\omega^2=1\,.$$
- "explication": in 1+1 dim. the SG theory has an infinite number of conserved quantities –a completely integrable Hamiltonian system

# Oscillons

- Perturbative computations of Dashen, Hasslacher, Neveu suggested the existence of breathers in the $1+1$ dimensional $\phi^4$-theory (1975)
- Numerical computations of the evolution of various Cauchy data
  $\rightarrow$ in non-linear (massive, $m$) scalar field theories meta-stable states (in $1, 2, 3$ dims.) with unusually long lifetimes $\tau \gg 1/m$ have been observed
  A. E. Kudryavtsev, I. L. Bogolyubskii, and V. G. Makhan'kov (1975-80) $\rightarrow$ called pulsons
- Collective effort of mathematicians and physicist $\rightarrow$ the unique $1+1$ dimensional non-trivial scalar theory admitting breathers is the SG one!
  Eleonski et al, (1984), Segur-Kruskal (1986), Vuillermot (1987), Kichenassamy (1991)

# Oscillons

- rediscovered by Gleiser '94,(re)named oscillons extensive ongoing studies : Adib, Gleiser, Müller, Watkins, Honda-Choptuik, Graham, Farhi, Tranberg-Saffin, Hindmarsh,...
- other (vector, tensor) fields may be present:
  coupling to electromagnetism, gauge-fields or gravitation
  also extension to higher spatial dimensions ($D \leq 7$)
- Nonlinearities are essential, and massive fields are necessary: $\omega < m$ modes are non-radiative
- $1+1$ or $1+2$ dimensions: infinitely many oscillations
  $1+3$ dimensions: sudden decay after about $10^3 - 10^4$ oscillations
- Oscillons have been found in the bosonic sector of the standard model N. Graham *Phys. Rev.Lett.* **98**, 101801 (2007)

# The uses of oscillons

- Oscillons are expected to have important effects on the dynamics of various systems (including the Early Universe), since they retain a considerable amount of energy.
- The persistence of oscillons in one spatial dimension in an expanding background metric has been observed
- They can easily form in physical processes:
  $\rightarrow$ QCD phase transition, where oscillon like objects in the axion field have been observed
  $\rightarrow$ in vortex-antivortex annihilation and in domain collapse
  $\rightarrow$ semiclassical decay of topological defects
- Once formed, oscillons could considerably influence the dynamics, e.g. in the case of the bubble nucleation process.

# Scalar theory in $1 + D$ dimensions

In this talk I shall concentrate on a (real) scalar field theory in $D$ spatial dimensions

$$-\frac{\partial^2\phi}{\partial t^2} + \Delta^{(D)}\phi = U'(\phi)\,, \quad \text{with} \quad \Delta^{(n)} = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}\,,$$

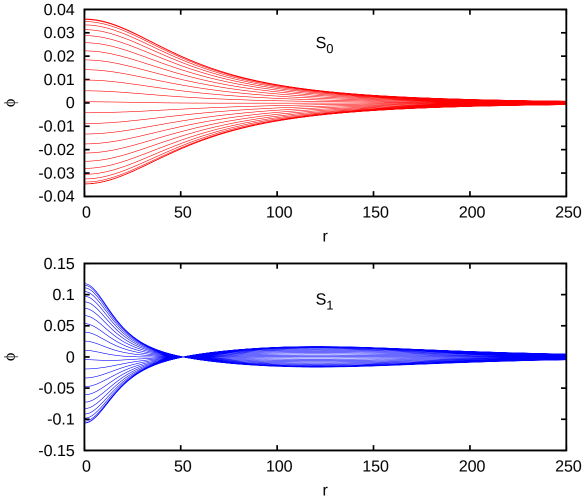
where $U(\phi)$ denotes the interaction potential.

minimum of potential is at $\phi = 0$, $\quad U'(0) = 0$
$\longrightarrow \quad \phi$ tends to zero at infinity

mass of small excitations around $\phi = 0$ is given by $U''(0) = m^2$

rescaling $t$ and the spatial coordinates by a constant we set $m = 1$

# Shape of a typical spherically symmetric oscillon in the $\phi^4$ model for D=3




solutions with nodes have higher energy ($E_0 \approx 632.5$, $E_1 \approx 4061.9$) and are less stable

# The 1+1 dimensional SG breather

sine-Gordon potential $U(\phi) = 1 - \cos\phi$

$$-\phi_{,tt} + \phi_{,xx} = \sin\phi$$

exponentially localized, time-periodic breather solution

$$\phi = 4\arctan\left[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\,\frac{\sin\left(\sqrt{1-\varepsilon^2}\,t\right)}{cosh\,(\varepsilon x)}\right]$$

solutions parametrized by frequency $\omega = \sqrt{1-\varepsilon^2} < 1 \quad (m=1)$

for small $\varepsilon$

- amplitude is proportional to $\varepsilon$
- characteristic size is proportional to $\frac{1}{\varepsilon}$

these properties remain valid for all small amplitude oscillons

# Scalar theory with a general potential

The SG potential is the only smooth one in 1+1 dimension for which exactly periodic localized breather solutions exist. However, for the signum-Gordon model breathers have been found, Arodź et al. (2008)

for other (continuous) potentials finite energy, generic initial data evolves into long living oscillating lumps – oscillons
– they lose energy by emitting small amplitude scalar waves
– their amplitude and frequency is changing very slowly
– for small amplitudes this change is very often so slow that it cannot be observed numerically

in $1 + D$ dimensions ($D \geq 2$) the sine-Gordon oscillons are not exactly periodic, nevertheless their radiation rate is suppressed $\rightarrow$ very long lifetimes

for a given $D$ exact breathers may exist for special potentials

# Exactly time periodic solutions

Time dependent oscillons can only be investigated presently by numerical evolution codes
– difficult, because massive fields produce high frequency oscillations moving out slower than the speed of light

Exactly time periodic solutions are easier to analyze – they can be calculated by solving the (truncated) Fourier mode equations numerically

$$\phi = \sum_{n=0}^{\infty} \phi_n \cos(n\,\omega t)$$

no $\sin(n\,\omega t)$ terms because of time reflection symmetry at $t = 0$

for spherical symmetry one obtains coupled ordinary differential equations for $\phi_n(r)$

# Mode decomposition

for large radius the modes decouple ($m = 1$ was set)

$$\Delta\phi_n = (1 - n^2\omega^2)\phi_n \ , \qquad \Delta = \frac{d^2}{dr^2} + \frac{D-1}{r}\frac{d}{dr}$$

asymptotically, $\phi_n$ with $n \geq n_0$ are oscillating in space
– if $\frac{1}{2} < \omega < 1$ then $n_0 = 2$

in order to have finite energy, the amplitudes of all these modes have to be zero

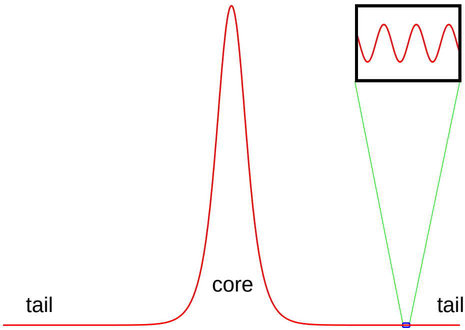
too many conditions to satisfy –generically, finite energy localized breathers are not expected to exist

There is, however, a plethora of time-periodic solutions, however they are not localized in space, and their total energy is divergent.

# Exactly periodic but only weakly localized solutions

energy loss of oscillons is compensated by small incoming radiation in order to make the solution exactly periodic

there are periodic solutions with very low energy density radiative tails
– called nanopterons by J. P. Boyd (*Nonlinearity* **3,** 177, 1990)



# Quasibreathers

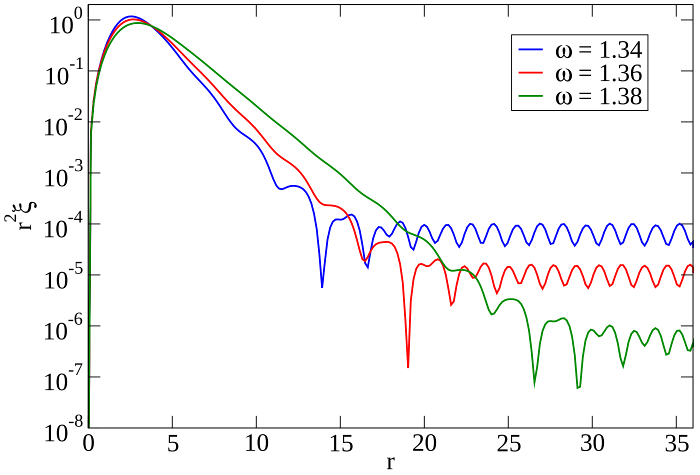
Look for weakly localized solutions whose oscillation amplitudes are so fine-tuned that the energy density of their radiative tail is minimized $\longrightarrow$ quasibreathers

intuitively a quasibreather is expected to give a good approximation for an oscillon of the same frequency

it has been found that such quasibreathers agree to a high precision with the same frequency oscillon in the core region and also in a substantial part of the radiative region (Fodor et al., 2006)

oscillons evolve adiabatically through quasibreather states

# Energy content of the radiative tails



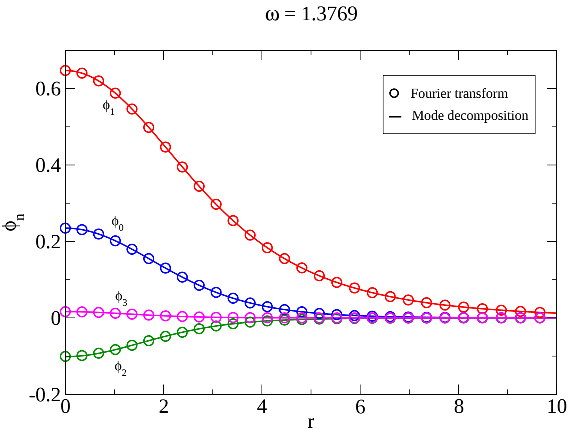
# Energy of quasibreathers

Total energy of quasibreathers is infinite – energy density of tail is small, but have to integrate it to infinite volume

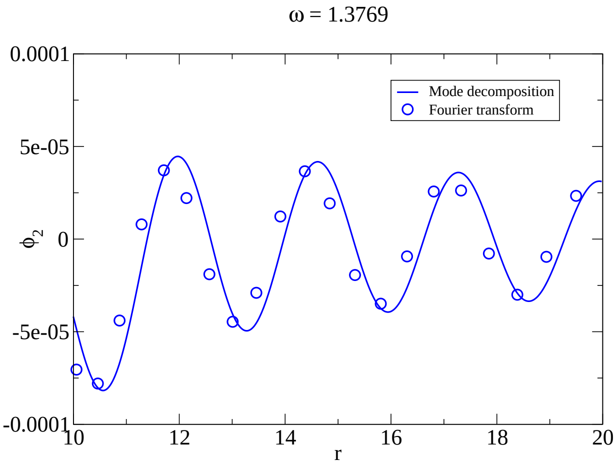
Energy of core $E(\omega)$ is a well defined function of the frequency (core ends where spatially oscillating modes start to dominate)

$E(\omega(t))$ describes the time dependence of the energy of oscillons very well

# Comparison of oscillons with quasibreathers



# Comparison with quasibreathers(2)




The agreement is quite good up to a phase.

# Small amplitude expansion of scalar theory

$1+1$ dimension: S. Kichenassamy,
*Comm. Pur. Appl. Math.* **44**, 789 (1991)

$$-\phi_{,tt} + \Delta^{(D)}\phi = U'(\phi) = \phi + \sum_{k=2}^{\infty} g_k \phi^k \,,$$

for the standard $\phi^4$ theory $U(\phi) = \frac{1}{8}\phi^2(\phi-2)^2$,
$g_2 = -\frac{3}{2}$, $g_3 = \frac{1}{2}$ and $g_i = 0$ for $i \geq 4$
for the sine-Gordon potential $U(\phi) = 1 - \cos(\phi)$,
$g_{2i} = 0$ and $g_{2i+1} = (-1)^i/(2i+1)!$.

Expand $\phi$ in terms of a small parameter $\varepsilon$ as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k$$

# Rescaled coordinates

Asymptotic behaviour of the leading mode $\phi_1$ shows that the size of configurations increases for small $\varepsilon$
We introduce rescaled spatial coordinates by

$$\zeta^i = \varepsilon x^i$$

One must also allow for the $\varepsilon$ dependence of the time-scale of the configurations, therefore a new time coordinate is introduced as

$$\tau = \omega(\varepsilon) t$$

$$\omega^2(\varepsilon) = 1 + \sum_{k=1}^{\infty} \varepsilon^k \omega_k$$

# Order $\varepsilon$

After these rescalings the field equation takes the form

$$-\omega^2\ddot{\phi} + \varepsilon^2\Delta\phi = \phi + \sum_{k=2}^{\infty} g_k\phi^k$$

– overdot means $\frac{\partial}{\partial\tau}$
– spatial derivatives are calculated with respect to $\zeta^i$

Substituting $\phi = \sum_{k=1}^{\infty}\varepsilon^k\phi_k$, the $\varepsilon$ order terms give

$$\ddot{\phi}_1 + \phi_1 = 0$$

$\phi_1 = p_1\cos(\tau+\alpha)$, where $p_1$ and $\alpha$ are functions of $\zeta^i$

$p_1$ and $\alpha$ will be determined at higher order

harmonic oscillator with frequency $\omega = 1$, fixed by $m = 1$

# Order $\varepsilon^2$

The $\varepsilon^2$ order terms give

$$\ddot{\phi}_2 + \phi_2 + g_2\phi_1^2 + \omega_1\ddot{\phi}_1 = 0\ , \quad \phi_1 = p_1\cos(\tau+\alpha)$$

the solution is

$$\begin{aligned}\phi_2 &= p_2\cos(\tau+\alpha) + q_2\sin(\tau+\alpha) + \frac{g_2}{6}p_1^2[\cos(2\tau+2\alpha) - 3] \\ &+\frac{\omega_1}{4}p_1\left[2\tau\sin(\tau+\alpha) + \cos(\tau+\alpha)\right]\end{aligned}$$

where $p_2$ and $q_2$ are some functions of $\zeta^i$

Since we are looking for solutions bounded in time, it is necessary to impose $\omega_1 = 0$ (then $\omega = 1 + \omega_2\varepsilon^2 + ...$)

absence of the resonance term $\omega_1\ddot{\phi}_1 = -\omega_1 p_1\cos(\tau+\alpha)$

## Order $\varepsilon^3$

The $\varepsilon^3$ terms give another forced oscillator equation

$$\begin{aligned}
&\ddot{\phi}_3 + \phi_3 + (p_1 \Delta\alpha + 2\nabla\alpha\nabla p_1)\sin(\tau+\alpha) \\
&- \left[\Delta p_1 + \omega_2 p_1 + \lambda p_1^3 - p_1(\nabla\alpha)^2\right]\cos(\tau+\alpha) \\
&+ \frac{1}{12} p_1^3 (2g_2^2 + 3g_3)\cos(3\tau + 3\alpha) \\
&+ g_2 p_1 \left[q_2 \sin(2\tau + 2\alpha) + p_2 \cos(2\tau + 2\alpha) + p_2\right] = 0
\end{aligned}$$

where $\lambda = \frac{5}{6} g_2^2 - \frac{3}{4} g_3$ has been introduced.

for $\phi^4$ theory $\lambda = 3/2$, for the sine-Gordon potential $\lambda = 1/8$

the coefficients of $\sin(\tau+\alpha)$ and $\cos(\tau+\alpha)$ must vanish in order to get solutions bounded in time

periodicity is a consequence of boundedness

# Constant phase

The vanishing of the coefficient of the $\sin(\tau + \alpha)$ term implies

$$\nabla(p_1^2 \nabla\alpha) = 0$$

from this

$$\int_\Omega \alpha \nabla(p_1^2 \nabla\alpha) = \int_{\partial\Omega} \alpha p_1^2 n \cdot \nabla\alpha - \int_\Omega p_1^2 (\nabla\alpha)^2 = 0$$

boundary term vanishes sufficiently fast $\longrightarrow \quad \nabla\alpha = 0$
$\alpha$ must be a constant which can be absorbed by a shift in $\tau$
from now on we set $\alpha = 0$
phase of oscillations is location independent

# $\varepsilon$ dependence of the frequency $\omega$

The vanishing of the $\cos\tau$ resonance term implies

$$\Delta p_1 + \omega_2 p_1 + \lambda p_1^3 = 0$$

– exponentially localized solutions exist only if $\omega_2 < 0$
  this means $\omega < 1$

– we set $\omega_2 = -1$ by a simultaneous rescaling of $\zeta^i$ and $p_1$

– the physical parameter determining the configuration is $\omega$

In general, it is possible to set

$$\omega = \sqrt{1 - \varepsilon^2}$$

# Master equation

$$\Delta p_1 - p_1 + \lambda p_1^3 = 0$$

multiplying by $p1$ and integrating $\longrightarrow \quad \lambda > 0$
defining $S = p_1\sqrt{\lambda}$, where $\lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3$ we get the master equation

$$\Delta S - S + S^3 = 0$$

– universal for the class of theories considered
– dependence on $U(\phi)$ enters only through $\lambda$
– By simple virial arguments it is readily shown that no exponentially localized solution of the master Eq. exist in dimensions $D < 4$

to lowest order

$$\phi = \varepsilon\phi_1 = \varepsilon p_1(\zeta^i)\cos(\tau) = \frac{\varepsilon S(\varepsilon x^i)}{\sqrt{\lambda}}\cos(\omega t)$$

where $\omega = \sqrt{1-\varepsilon^2}$

# Higher orders

$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k$, where

$$\begin{aligned}
\phi_1 &= p_1 \cos\tau \\
\phi_2 &= \frac{1}{6} g_2 p_1^2 \left(\cos(2\tau) - 3\right) \\
\phi_3 &= p_3 \cos\tau + \frac{1}{72}(4g_2^2 - 3\lambda) p_1^3 \cos(3\tau)
\end{aligned}$$

– can be continued, but expressions become longer
– no $\sin(k\tau)$ terms $\longrightarrow$ time reflection symmetry
– odd $\phi_i$ contains only odd Fourier modes, even only even
– $p_3$ is determined by a linear differential equation containing source terms nonlinear in $p_1$
– for potentials symmetric around the minimum (sine-Gordon) $g_{2i} = 0 \longrightarrow \phi_{2i} = 0$, only odd Fourier modes

# Asymptotic series

If exponentially decreasing $S$ exist, all $\phi_n$ are also exponentially localized

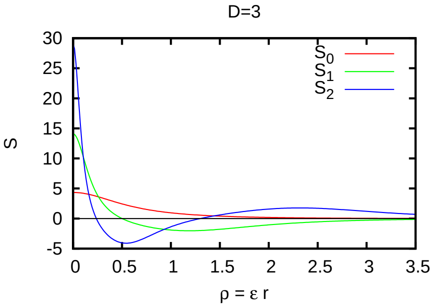
The series solution in powers of $\varepsilon$ does not converge to a breather, it is an asymptotic series
– H. Segur and M. D. Kruskal, *Phys. Rev. Lett.* **58**, 747 (1987) for $1+1$ dimension

It corresponds to a quasibreather whose standing wave tail is smaller than $\varepsilon^n$ for any $n > 0$ (beyond all orders)
tail amplitude is exponentially small $\sim \exp(-C/\varepsilon)$

To a given order in the expansion, for sufficiently small values of $\varepsilon$ the corresponding sum yields an excellent approximation to the core part of an oscillon.

# Solutions of the master equation

In case of spherical symmetry a discrete family of localized solutions exist for $1 < D < 4$, indexed by the number of nodes




solutions with nodes correspond to higher energy less stable oscillons

# Energy of the quasibreathers in $D$ dimensions

In the rescaled coordinate system, $\tau\,,\zeta$, the energy

$$E = \frac{1}{\varepsilon^D}\int d^D\zeta\,\mathcal{E}\,,\quad \mathcal{E} = \frac{1}{2}(1-\varepsilon^2)\,(\partial_\tau\phi)^2 + \varepsilon^2\frac{1}{2}\,(\partial_i\phi)^2 + U(\phi)\,.$$

The time-averaged energy density, $\bar{E}$, is easily computed in the $\varepsilon$ expansion: for the $\phi^4$ theory:

$$\bar{\mathcal{E}} = \frac{\varepsilon^2}{3}S^2 + \varepsilon^4\left[\frac{1}{6}(\nabla S)^2 - \frac{41}{108}S^2(S^2+2) + \frac{65}{36}SZ\right]S^4\,.$$

Therefore

$$\bar{E} = \varepsilon^{2-D}\frac{E_0}{2\lambda} + \varepsilon^{4-D}E_1\,,\quad \text{where}\quad E_0 = \int d^D\zeta\,S^2\,,$$

$E_1$ denoting the integral of the 4-th order term.

# Energy of the quasibreathers in $D$ dimensions

The leading order behaviour of the energy: $\bar{E} \propto \varepsilon^{2-D}$.

In $D > 2$ the total energy *increases* without any bound for decreasing values of $\varepsilon$.

In $D = 2$ the (averaged) energy $\bar{E} \to \mathrm{const.}$
while in $D < 2$ $\bar{E} \to 0$ as $\varepsilon \to 0$.

In dimensions $D > 2$ the core energy of a QB exhibits a minimum for some frequency $\omega_{\mathrm{m}}$:

$$\omega_{\mathrm{m}}^2 = 1 - \varepsilon_{\mathrm{m}}^2 = 1 - \frac{1}{2\lambda} \frac{(D-2)E_0}{(4-D)E_1} .$$

$\to$ for $D > 2$ instability for $\omega > \omega_{\mathrm{m}}$

# Energy of the quasibreathers in $D$ dimensions

In $D = 1$ $S(\zeta) = \sqrt{2}\mathrm{sech}(\zeta)$, $\to E_0 = 4$, and $\bar{E} = 2\varepsilon/\lambda + \mathcal{O}(\varepsilon^3)$.

In $D = 2$ , and $D = 3$ the first two terms of the energy:

$$\bar{E} \approx 3.9003 + 26.9618\varepsilon^2 \,, \quad \text{for } D = 2 \,,$$

$$\bar{E} \approx 6.29908/\varepsilon + 264.262\varepsilon \,, \quad \text{for } D = 3 \,.$$

This simple estimate gives $\varepsilon_{\mathrm{m}} \approx 0.15428$,

For spherically symmetric oscillons in $D = 3$ we have measured $\omega_{\mathrm{m}} \approx 0.9659$, $\to \varepsilon_{\mathrm{m}} \approx 0.2588$.

# Conclusions

- In massive scalar field theories generic Cauchy data evolves into almost time-periodic, spatially localized configurations (oscillons)
- Small amplitude oscillons are very well described by an asymptotic series of localized quasibreathers
- Small amplitude scalar oscillons appear to be very stable in $D = 1, 2$, in $D = 3$ they have at least 1 unstable mode
- Scalar oscillons appear to be stable in the presence of other fields (including zeromass photons).
- No exponentially localized small amplitude QB's exist in dimensions $D \geq 4$
- Radiative tail is exponentially small in terms of the amplitude parameter