VI.RELATIVISTIČKE JEDNADŽBE

(FEČ § 2.3.2 str. 81)

- PRIKAZ ČESTICA LOKALNIM POLJIMA
- KLEIN-GORDONOVA JEDN.
- DIRACOVA JEDN.

REPREZENTACIJE LORENTZOVIH TRANSFORMACIJA

Lorentzove transf. (HLG) $\quad x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$

rotacije i potisci $\in L^{\uparrow}_{+}$ $\quad$ ortokrone, $\Lambda^0{}_0 \geq 1$; vlastite, $\det\Lambda = +1$

reprezentirane s $\quad U(\Lambda) = e^{-\frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma}}$

$$M^{\mu\nu} = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & J_3 & -J_2 \\ -K_2 & -J_3 & 0 & J_1 \\ -K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$$\psi' = U(\Lambda)\psi$$

$$\epsilon_{kij} M^{ij} = J_k$$

$$M^{0i} = K_i$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k$$

The Lorentz group

Rotations J_i Boosts K_i

$$[J_i, J_j] = i\varepsilon_{ijk} J_k$$
$$[J_i, K_j] = i\varepsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\varepsilon_{ijk} J_k$$

} Generate the group SO(3,1)

$$(M_{\rho\sigma} = i(x_\rho \tfrac{\partial}{\partial x^\sigma} - x_\sigma \tfrac{\partial}{\partial x^\rho}) \qquad J_i = \tfrac{1}{2}\varepsilon_{ijk} M_{jk} \qquad K_i = M_{0i})$$

To construct representations a more convenient (non-Hermitian) basis is

$$N_i = \tfrac{1}{2}(J_i + iK_i)$$

$$[N_i, N_j] = i\varepsilon_{ijk} N_k$$
$$[N_i^\dagger, N_j^\dagger] = i\varepsilon_{ijk} N_k^\dagger$$
$$[N_i, N_j^\dagger] = 0$$

Representations $J_i = N_i + N_i^\dagger$

$(n, m) \quad J = n + m$

$(0,0)$ scalar $J=0$

$(\tfrac{1}{2}, 0), (0, \tfrac{1}{2})$ LH and RH spinors $J=\tfrac{1}{2}$

$(\tfrac{1}{2}, \tfrac{1}{2})$ vector $J=1$, etc

& POINCAREOVIH TRANSF.

Poincaré'ove transf. (NLG)

$$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$$

infinitez.

$$U(\Lambda, a) = I - \frac{i}{2}\omega_{\rho\sigma}M^{\rho\sigma} + i a_\mu P^\mu$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\left(g^{\mu\rho}M^{\sigma\nu} - \dots\right)$$

$$[M^{\mu\nu}, P^\rho] = -i g^{\mu\rho}P^\nu + i g^{\nu\rho}P^\mu$$

$$[P^\mu, P^\rho] = 0$$

Poincaré-ova algebra

1. Casimirova invarijanta : $P^2 = P^\mu P_\mu$

$$[P^2, M^{\mu\nu}] = 0$$

2. Casimirova invarijanta : $W^2 = W^\mu W_\mu$

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}$$

- za masivne č. $W^2 = -\vec{W}^2 = -m^2 s(s+1)$
- za bezmasene č. $W_\mu|p\rangle = \lambda P_\mu|p\rangle$; $\lambda = \frac{\vec{P}\cdot\vec{J}}{P^0}$ = helicitet

CASIMIROVE INVARIJANTE

Katalogizacija stanja prema

$P^2 = m^2$	P^0 (syn. inv. na Lor. transf.)	
>0	>0	$W^2 = -m^2 s(s+1)$
>0	<0	
$=0$	>0	$W^2 = 0$ $W^2 = -\rho^2$ (kontin)
$=0$	<0	
<0	virtualne č, prostornog imp.	
$P^\mu \equiv 0$	vakuum	

PRIKAZ ČESTICA LOKALNIM POLJIMA

ČESTICU OPISUJEMO FUNKCIJOM KOJA SE NE MIJENJA PRI TRANSLACIJAMA U PROSTORU I VREMENU (LOKALNIM POLJEM):

$$x^\mu \to x'^\mu = x^\mu + \underbrace{\delta x^\mu}_{= \varepsilon^\mu} \Rightarrow \phi(x) \to \phi'(x) = \phi(x) + \delta_0 \phi(x)$$

Za transl. $\delta_0 \phi = \phi'(x) - \phi(x) = -\varepsilon^\mu \partial_\mu \phi = -i \varepsilon^\mu P_\mu \phi$

$$\delta\phi = \phi'(x') - \phi(x)$$

$$= \underbrace{\phi'(x') - \phi(x')}_{\delta_0 \phi} + \phi(x') - \phi(x)$$

$$\phi(x') = \phi(x) + \delta x^\mu \partial_\mu \phi$$

$$= \delta_0 \phi + \delta x^\mu \partial_\mu \phi$$

ε^μ za translacije

$$\delta f = 0 = \delta_0 f + \varepsilon^\mu \partial_\mu f \Rightarrow \delta_0$$

TENZORSKA POLJA

Lokalna polja

invarijantna na translacije ($\in$ NLG)

tenzorska polja (HLG)

$(2S+1)$ – komponentna za spin "S"

SKALARNA $\phi(x) \rightarrow \phi'(x') = \phi(x)$ (tj. $\phi'(x) = \phi(\Lambda^{-1}x)$)

SPINORNA $\psi(x) \rightarrow \psi'(x') = \underbrace{S(\Lambda)}\psi(x)$ treba naći!

VEKTORSKA $A^{\mu}(x) \rightarrow A'^{\mu}(x') = \Lambda^{\mu}{}_{\nu} A^{\nu}(x)$

rep. Lor. gr. od generatora grupe!

TENZORSKA $U_{\mu\nu}(x) \rightarrow U'_{\mu\nu}(x') = \underbrace{\Lambda_{\mu\rho}\Lambda_{\nu\sigma}}_{= T_{\mu\nu,\rho\sigma}} U^{\rho\sigma}(x)$

reducibilna

$U_{\mu\nu} = \underset{[10\text{-d}]}{U^{S}_{\mu\nu}} + \underset{[6\text{-d}]}{U^{A}_{\mu\nu}}$ – komp. S & A-tenzora transf. se na Λ neovisno, bez miješanja

RELATIVISTIČKE INVARIJANTE

- KOMBINACIJE (PRODUKTI) KOJI SE NE MIJENJAJU PRI LORENTZOVIM TRANSFORMACIJAMA

$\phi(x)\,\phi(x)$ skalarnih polja

$A_\mu(x)\,A^\mu(x)$ vektorskih - " -

$\bar{\psi}(x)\,\gamma_\mu\,\partial^\mu\,\psi(x)$ spinornih - " -

$\bar{\psi}(x)\,\gamma_\mu\,\psi(x)\,A^\mu(x)$ miješanih

$\bar{\psi}(x)\,\psi(x)\,\phi(x)$

RELATIVISTIČKI OBLIK MAXWELL-ove TEORIJE

u racionaliziranim Gaussovim (Heaviside-Lorentz-ovim) jedinicama 6 polja & 4 izvora

$(\vec{E}, \vec{B}) \qquad (\rho, \vec{j}) = j^\mu$

zadov.

$$\nabla\cdot\vec{E} = \rho \qquad \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{B} = 0 \qquad \nabla\times\vec{B} = \vec{j} + \frac{\partial\vec{E}}{\partial t}$$

– prijelazom na potencijale

$$A^\mu = (A^0, \vec{A}) \qquad \begin{cases} \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}, & A^0 = \phi \\ \vec{B} = \nabla\times\vec{A} \end{cases}$$

i antisim. tenzor (sa 6 komp.!), $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}$$

$$\partial_\mu F^{\mu\nu} = j^\nu$$

Uz "dualni" tenzor

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

$$\mathcal{L}_{Maxw.} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - j^\mu A_\mu$$

Uočimo da su prijelazom na $A^\mu = (\phi, \vec{A})$

- – dvije Maxw. j-be automatski zadov. $\nabla(\nabla\times\vec{A}) \equiv 0 \Rightarrow \nabla\vec{B} = 0$; $\nabla\times\nabla\phi \equiv 0 \Rightarrow \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$
- – preostale dvije $\partial_\mu F^{\mu\nu} = j^\nu$

$$\Box A^\nu - \partial^\nu(\underbrace{\partial_\mu A^\mu}_{=0}) = j^\nu$$

Odabirom *) Lorentzovog uvjeta ("baždarenja")

$\Box A^\nu = 0 \Rightarrow$ K.G. j-ba za slobodno polje ($j^\nu = 0$)

*) sloboda baždarne transf.

$$\left.\begin{aligned} \phi &\to \phi + \frac{\partial\chi(x)}{\partial t} \\ \vec{A} &\to \vec{A} - \nabla\chi(x) \end{aligned}\right\} A^\mu \to A^\mu + \partial^\mu\chi(x)$$

ostavlja $\vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}$

$\vec{B} = \nabla\times\vec{A}$

nepromijenjenima!

RELATIVISTIČKE JEDNADŽBE

$\hbar = c = 1$

Klein-Gordon

$$\boxed{E^2 = \vec{p}^{\,2} + m^2} \Rightarrow E_0 = m$$

$$-\frac{\partial^2}{\partial t^2}\phi = (-\nabla^2 + m^2)\phi \iff (\Box + m^2)\phi = 0$$

$$\frac{\partial}{\partial t}\left[i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right)\right] +$$

$$+ \nabla \cdot \left[-i(\phi^* \nabla \phi - \phi \nabla \phi^*)\right] = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \vec{j} = 0 \qquad \partial_\mu j^\mu = 0$$

$$E < 0 \leftrightarrow \rho < 0$$

LINEARIZACIJA KG JEDN.

$$i\frac{\partial\psi(\vec{x},t)}{\partial t} = (-i\vec{\alpha}\cdot\nabla + \beta m)\psi \equiv H\psi$$

$$\begin{aligned}\left(i\frac{\partial}{\partial t}\right)^2\psi &= (-i\vec{\alpha}\cdot\nabla + \beta m)(-i\vec{\alpha}\cdot\nabla + \beta m)\psi \\ &= \Bigg[-\sum_{i=1}^{3}\alpha_i^2\frac{\partial^2}{(\partial x^i)^2} - \sum_{i>j=1}^{3}(\alpha_i\alpha_j + \alpha_j\alpha_i)\frac{\partial^2}{\partial x^i\partial x^j} \\ &\quad -im\sum_{i=1}^{3}(\alpha_i\beta + \beta\alpha_i)\frac{\partial}{\partial x^i} + \beta^2 m^2\Bigg]\psi\ .\end{aligned}$$

KAO LINEARIZIRANA KLEIN-GORDONOVA, DAJE POZITIVNU GUSTOĆU VJEROJATNOSTI

Dirac

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = (-i\vec{\alpha}\cdot\nabla + \beta m)\psi(\vec{x},t) \iff (i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$\frac{\partial}{\partial t}\underbrace{[\psi^\dagger\psi]}_{\underline{\rho>0}} + \nabla[\psi^\dagger\vec{\alpha}\psi] = 0 \qquad \partial_\mu j^\mu = 0$$

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

KOVARIJANTNI ZAPIS I DIRACOVE MATRICE

$$i\frac{\partial \Psi}{\partial t} = H_D \Psi \quad ; \qquad H_D = -i\vec{\alpha}\cdot\nabla + \beta m$$

$$\{\alpha_i, \alpha_j\} = 0 \quad i \neq j$$

$$\{\alpha_i, \beta\} = 0 \quad , \quad \alpha_i^2 = \beta^2 = \mathbb{1}$$

$$\gamma^\mu = (\gamma^0, \gamma^i) \quad , \quad \underbrace{\gamma^0 = \beta \quad , \quad \gamma^i = \beta\alpha^i}$$

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0$$

$$\dim \begin{cases} \text{parna} \\ \geq 4 \end{cases}$$

DIRACOVA REPREZENTACIJA GAMA MATRICA

$$\gamma^0 = \beta = \sigma^3 \otimes I = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

$$\vec{\gamma} = \beta \vec{\alpha} = i\sigma^2 \otimes \vec{\sigma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

(4×4) matrice

$\Rightarrow \quad \psi$ - 4-komponentni

DIRACOVA INTERPRETACIJA NEGATIVNIH ENERGIJA

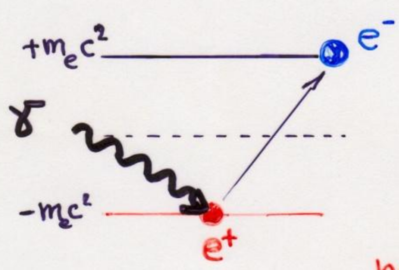
Razlog postojanja antičestica

– u spoju KVANTNE MEHANIKE i SPECIJALNE TEORIJE RELATIVNOSTI

Diracova j-ba za elektron

$$\left(i\frac{\partial}{\partial t} + i\vec{\alpha}\cdot\vec{\nabla} - \beta m_e\right)\Psi_e = 0$$

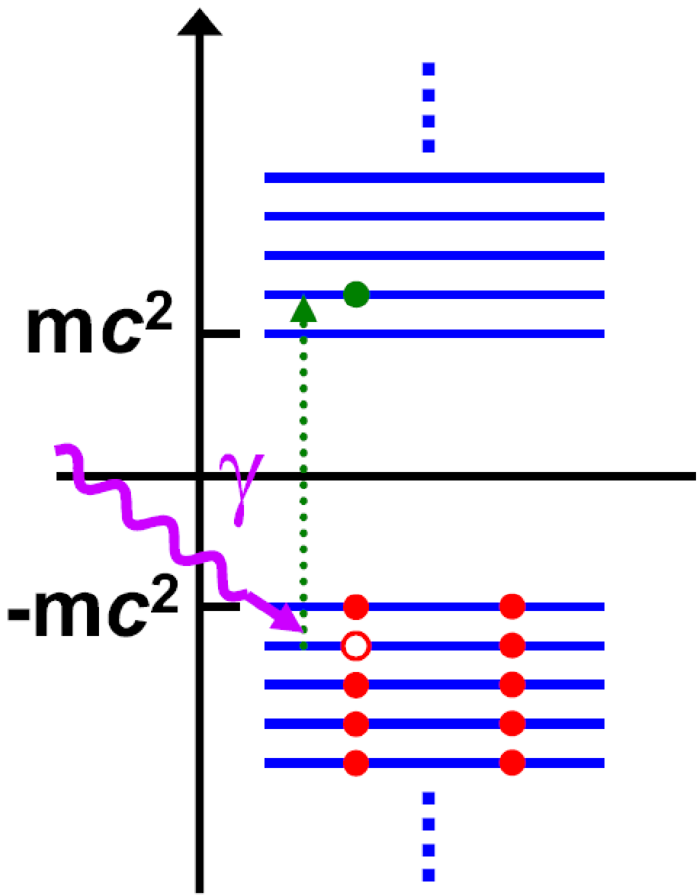
predviđa da njeno rješenje Ψ_e istodobno opisuje pozitron



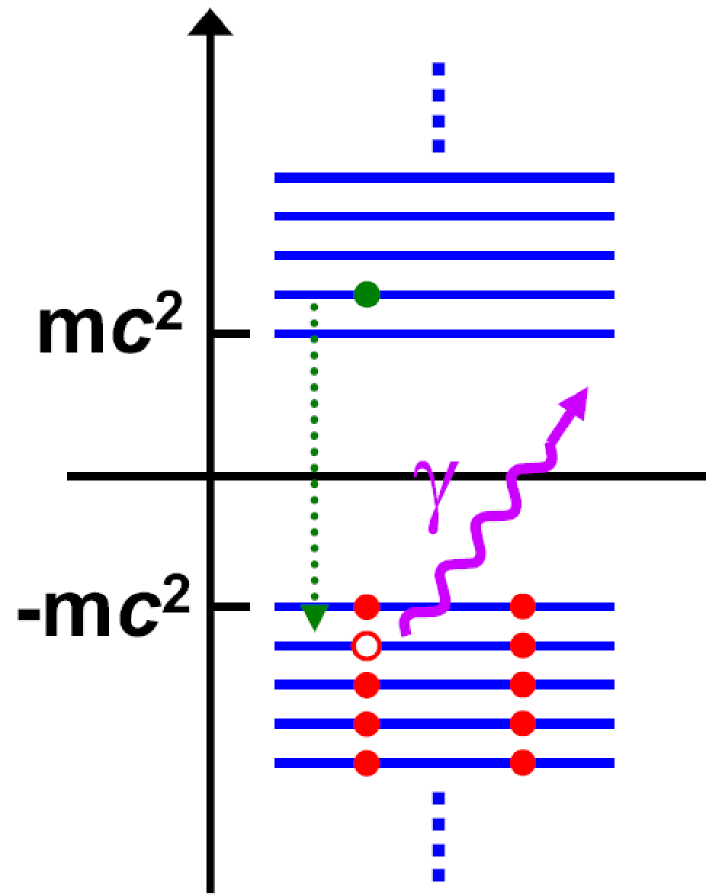
"rupa" u negativnoenergijskom "Diracovom moru" ponaša se kao "antielektron"

POPUNJENO MORE NEGATIVNIH ENERGIJA

$$\gamma \to e^- e^+$$

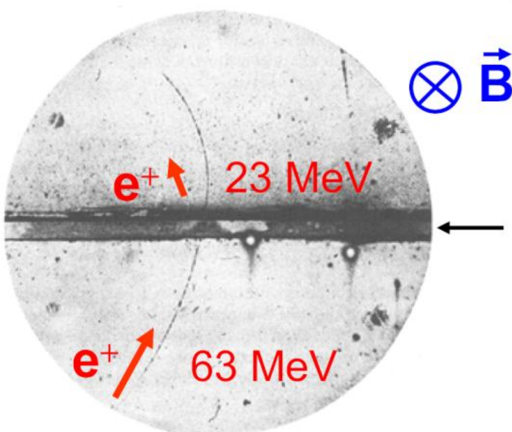


$$e^- e^+ \to \gamma$$

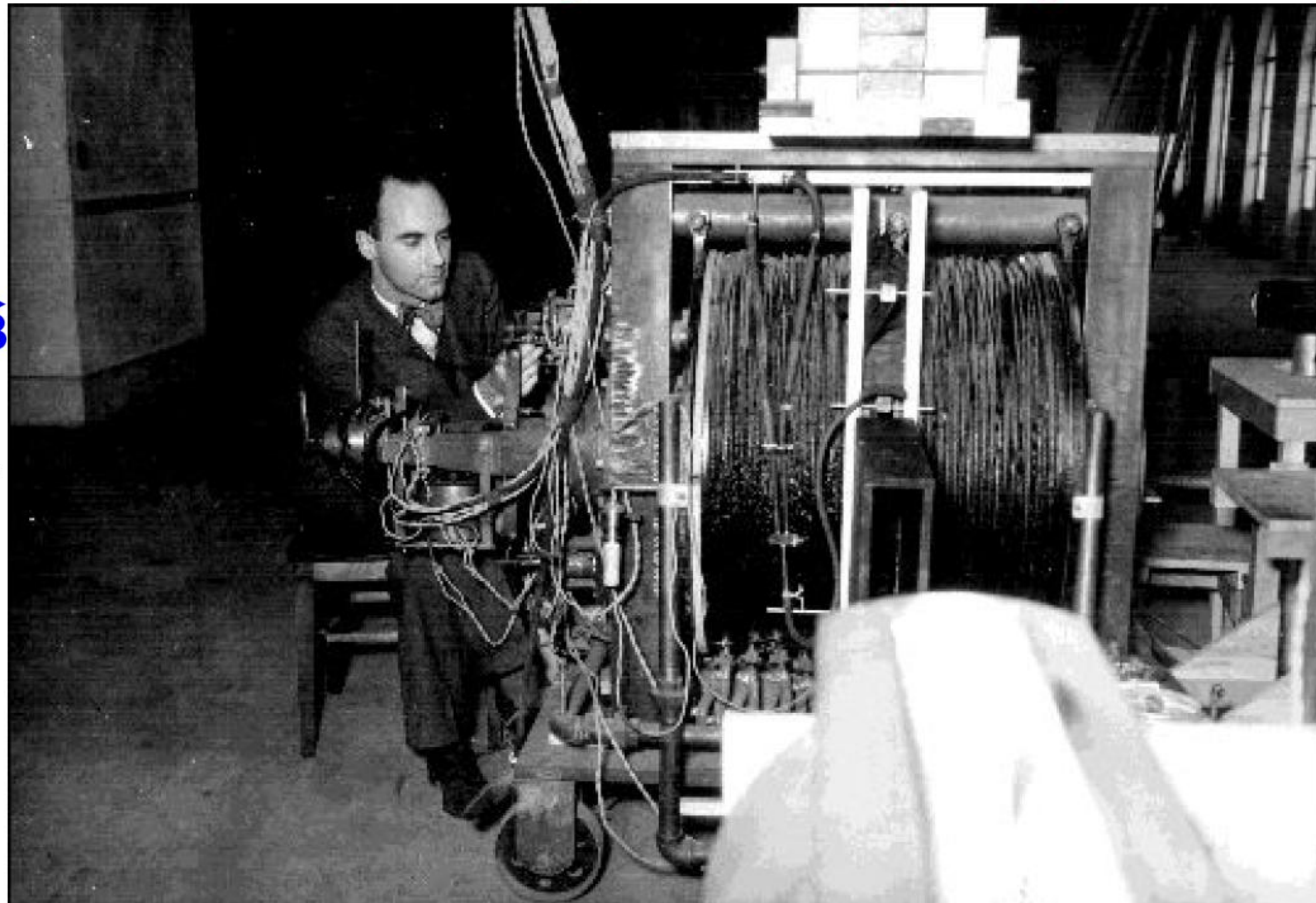


OTKRIĆE POZITRONA

C.D.Anderson, Phys Rev 43 (1933) 491



6 mm olovna ploča



DIRACOVA JEDNADŽBA

CPT simetrija:

Diracova jednadžba $\quad e^- \xleftarrow{\text{CPT}}\!\!\!\xrightarrow[\text{partneri}]{} e^+$

$$i\frac{\partial \Psi(\vec{x},t)}{\partial t} = (-i\vec{\alpha}\cdot\nabla + \beta m)\Psi$$

$$\left.\begin{array}{l} \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \\ \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \end{array}\right\} \Rightarrow \gamma^\mu = (\gamma^0, \vec{\gamma})$$

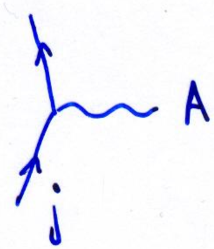
$$\gamma^0 = \beta, \quad \vec{\gamma} = \beta\vec{\alpha}$$

$$\boxed{(i\gamma^\mu\partial_\mu - m)\Psi = 0}$$

$$\mathcal{L}_D = \bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi$$

$$\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$$

$$\mathcal{L}_{int} = e\bar{\Psi}\gamma_\mu\Psi A^\mu = j_\mu \cdot A^\mu$$



$$H_{int} = -\int d^3x\, \mathcal{L}_{int}$$

RJEŠENJA DIRACOVE JEDNADŽBE FEČ str. 137

$$\psi^{(1)} = e^{-i(E/\hbar)t} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Electron (spin up)

$$\psi^{(2)} = e^{-i(E/\hbar)t} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Electron (spin down)

$$\psi^{(3)} = e^{+i(E/\hbar)t} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

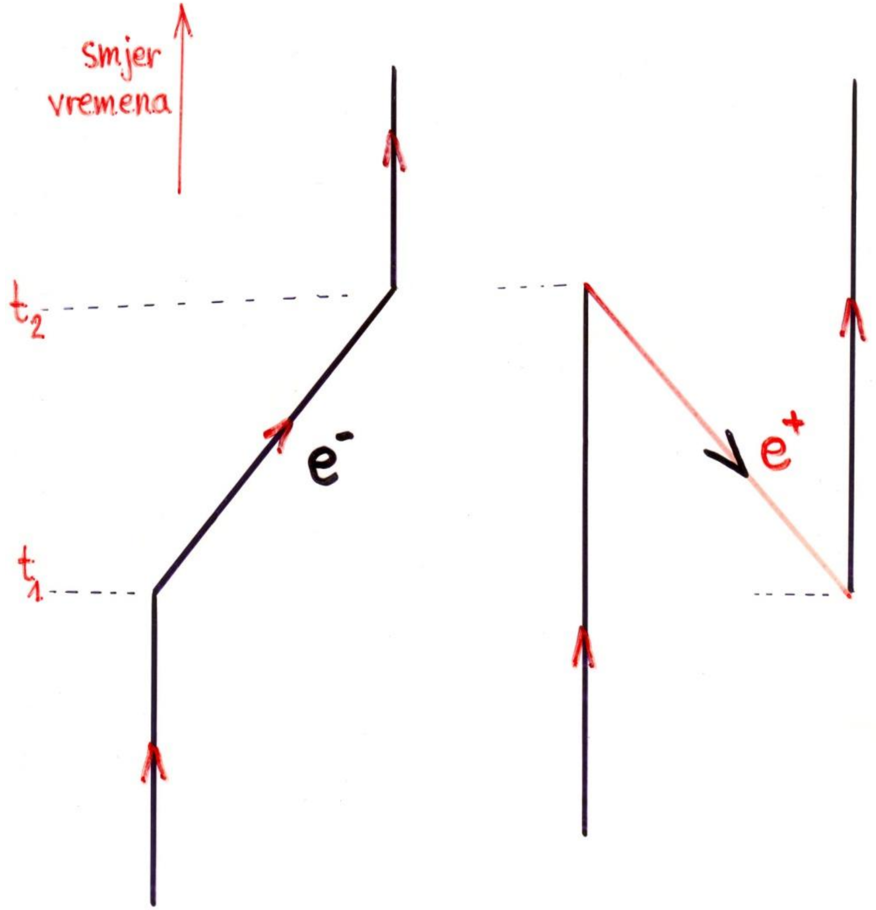
Positron (spin up)

$$\psi^{(4)} = e^{+i(E/\hbar)t} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Positron (spin down)

R.P. FEYNMAN

Dva viđenja istog događaja "dvostrukog raspršenja":

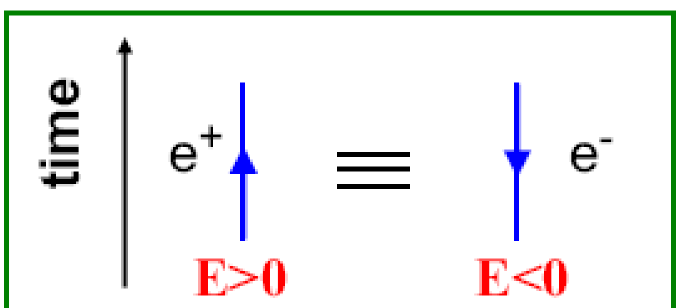


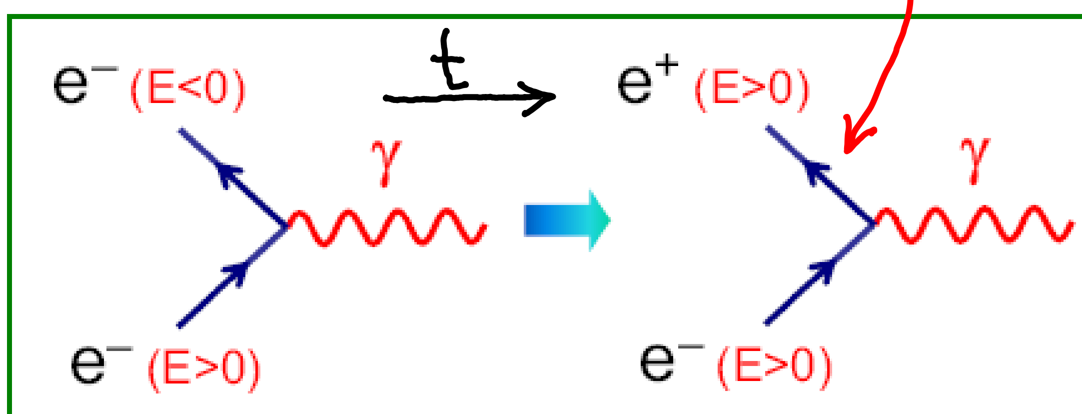
(a) elektron "putuje" između dva raspršenja

(b) pozitron i dva elektrona "putuju" između STVARANJA PARA i PONIŠTENJA PARA

FEYNMAN-STUECKELBERGova INTERPRETACIJA rješenja negativnih energija:

- ČESTICE NEGATIVNIH ENERGIJA KOJE PUTUJU U PROŠLOST
- ANTIČESTICE POZITIVNIH ENERGIJA KOJE PUTUJU U BUDUĆNOST – strelica protivno vremenu označavat će AČ rješenje:





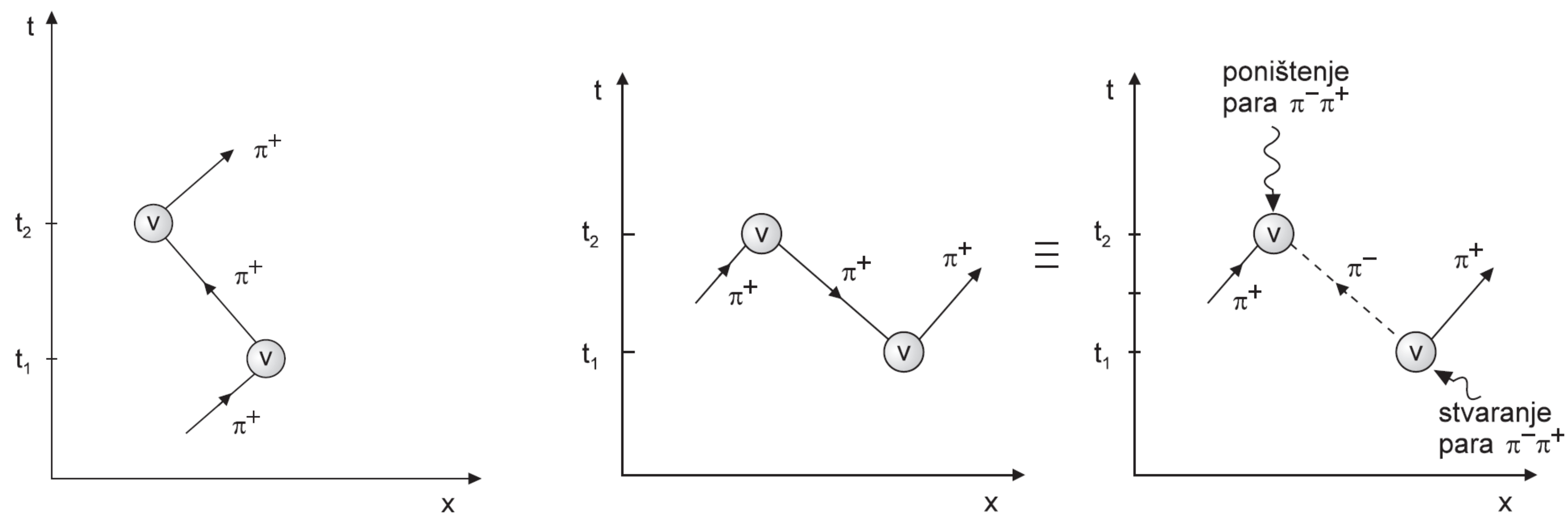
F-S INTERPRETACIJA RADI I ZA BOZONE (dodatna staza piona)

Feynman – Stuckelberg interpretation

$$\pi^+(E>0) \equiv \pi^-(E<0)$$

t

$$e^{-iEt} \qquad e^{-i(-E)(-t)}$$



UVODI NOVU STAZU (QM AMPLITUDU)

Two different time orderings giving same observable event :