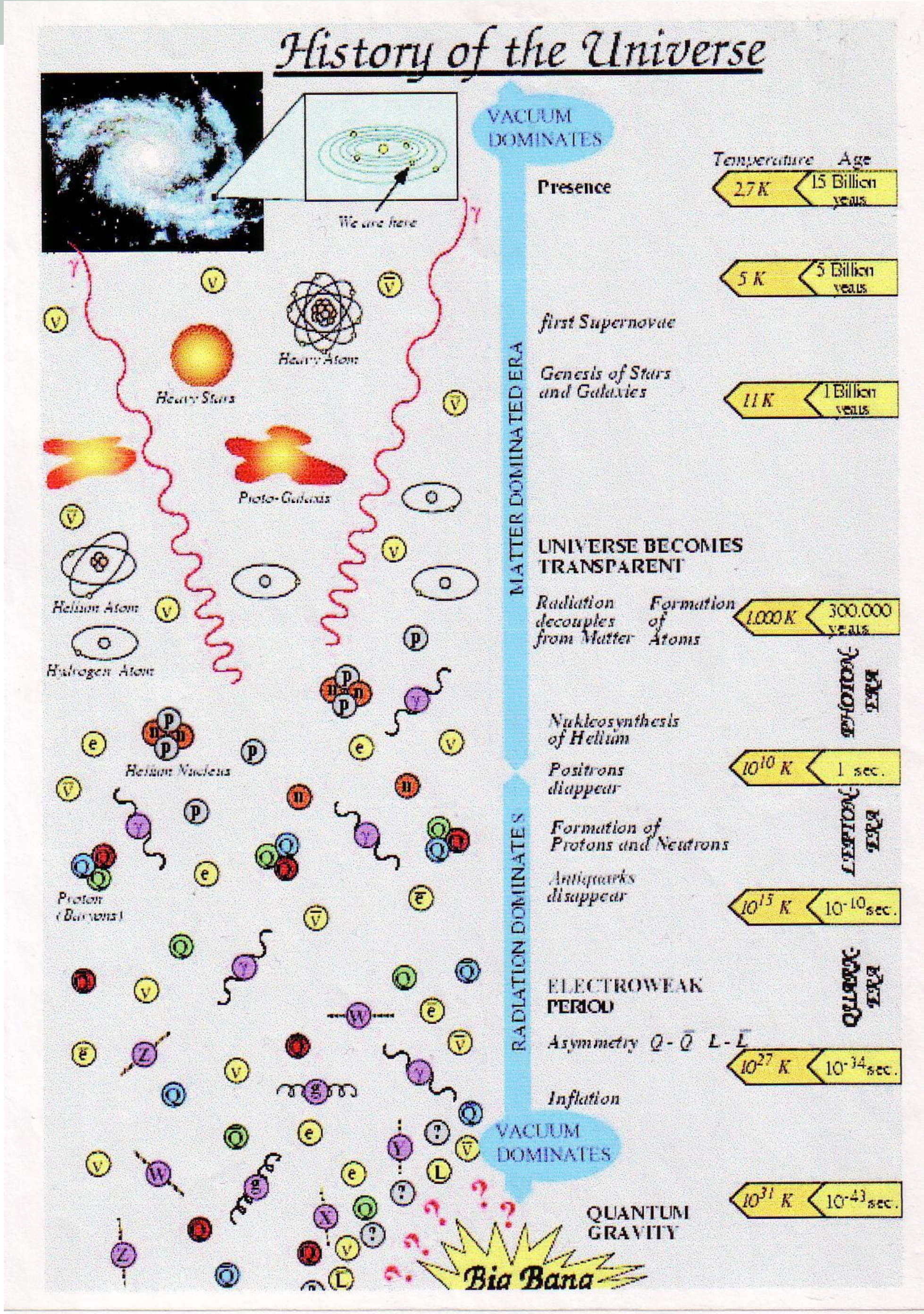
FIZIKALNA

KOZMOLOGIJA

III. FRIEDMANNOVE JEDNADŽBE ZA IDEALIZIRANI SVEMIR



Einsteinova kozmološka jednadžba

Od OTR do KOZMOLOGIJE

Narlikar '93, Ch 3

◊ Einsteinov svemir (1917)

- zamišljen kao posvuda ispunjen materijom
  uz dodatne pretpostavke SIMETRIJA:
- HOMOGENOST & IZOTROPNOST

  uz ondašnje vjerovanje astronoma da nema širenja ili sažimanja, pretpostavlja
- STATIČNOST

  → omogućava izoliranje vremenske koordinate:

$$ds^2 = \underset{\uparrow}{c^2} dt^2 - \alpha_{ij} dx^i dx^j \qquad (i,j = 1,2,3)$$

konst. (zbog homogenosti)

$\nexists\ dt\,dx^i$ (zbog izotropnosti)

Nadalje, Einsteinov izbor 3-D prostora,

- ZATVORENOG

  vodi na odabir 3-D površine S_3 (površine 4-D hipersfere radijusa S) dane j-bom (u Kartezijevim koor.)

$$(x_1)^2 + (x_2)^2 + (x_3)^2 + (x_4)^2 = S^2$$

[Narlikar §3.3 $k=+1$
ibid. §3.4 za negativnu $(k=-1)$ ili iščezavajuće zakrivlj.]

Metrika idealnog širećeg svemira

- Friedmann-Lemaître-Robertson-Walker (FLRW) metric : metric for a spatially homogeneous and isotropic expanding universe, with scale/expansion factor a(t) and curvature k

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right)$$

expansion factored out

$H(t) \equiv \frac{\dot{a}(t)}{a(t)}$ expansion rate or Hubble parameter

$H_0 \equiv H(t_0) \quad t_0 = today$

r,θ,φ spherical comoving coordinates:

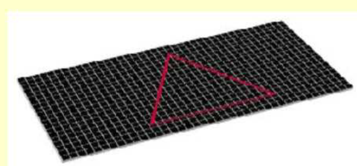
r : dimensionless & stationary wrt expansion

k, gaussian curvature:

closed universe

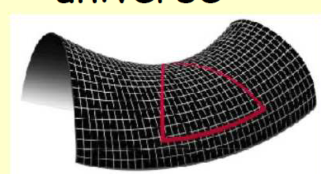
k=+1

flat universe



k=0

open universe



k=-1

Vježba 2.3: Hubbleov zakon u FLRW metrici

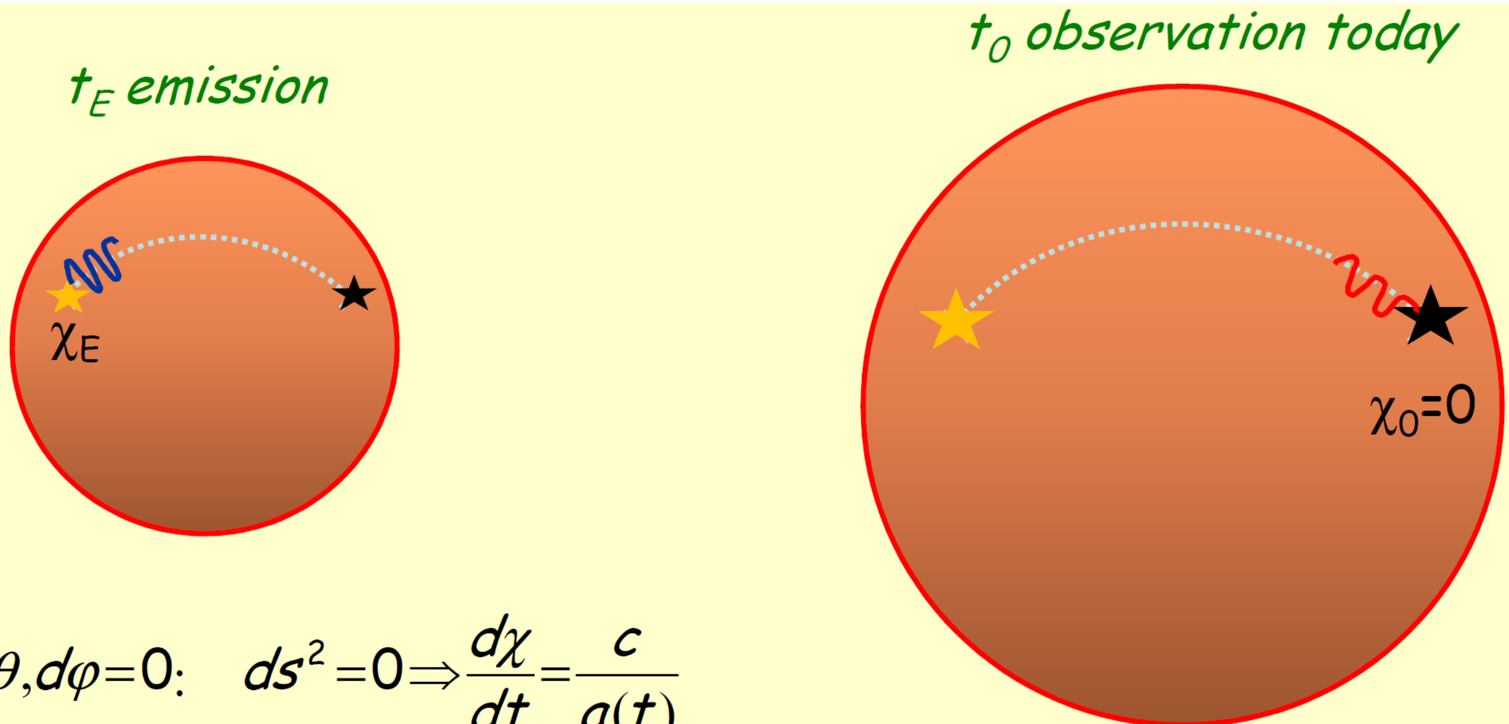
- FLRW metric: $$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right)$$
$$ds^2 = c^2 dt^2 - a(t)^2 \left(d\chi^2 + r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right) \right)$$
$$r = f(\chi) = \begin{cases} \sin\chi & k=1 \\ \chi & k=0 \\ sh\chi & k=-1 \end{cases}$$

- Hubble's law:
$$D(t) = a(t)\Delta\chi \Rightarrow \frac{dD}{dt} = \dot{a}(t)\Delta\chi = \frac{\dot{a}(t)}{a(t)} D(t) \Rightarrow \frac{dD}{dt} = H(t)D(t)$$

- Particle trajectories :

$ds^2 = 0$ geodesic equation (shortest path in three-space and maximum proper time)

Vježba 2.4:
Crveni pomak pri širenju fotona



$$d\theta, d\varphi = 0: \quad ds^2 = 0 \Rightarrow \frac{d\chi}{dt} = \frac{c}{a(t)}$$

$$\chi_E = \int_{t_E}^{t_0} \frac{cdt}{a(t)} = \int_{t_E+\lambda_E/c}^{t_0+\lambda_0/c} \frac{cdt}{a(t)} = \int_{t_E+\lambda_E/c}^{t_E} \frac{cdt}{a(t)} + \int_{t_E}^{t_0} \frac{cdt}{a(t)} + \int_{t_0}^{t_0+\lambda_0/c} \frac{cdt}{a(t)} = \int_{t_E}^{t_0} \frac{cdt}{a(t)} + \frac{\lambda_0}{a_0} - \frac{\lambda_E}{a_E}$$

$$\frac{\lambda_0}{\lambda_{emitted}} \equiv 1 + z = \frac{a_0}{a(t_{emission})}$$

light from distant sources is redshifted

Einsteinove jednadžbe

(polazeći od Riemannovog tenzora zakrivljenosti)

- **10 jedn. polja**
- **4 jedn. gibanja**
- **Usporedba geodetske jednadžbe i Lorentzove sile**

Tenzori nižeg ranga

Ricci-jev $R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\nu\sigma} \equiv R^{\sigma}{}_{\mu\nu\sigma}$

simetričan $R_{\mu\nu} = R_{\nu\mu}$;

Skalarna zakrivljenost

$$R = g^{\mu\nu} R_{\mu\nu} \;;$$

Einsteinov tenzor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

s posebnom ulogom u Einsteinovoj **OTR:**

zakrivljenost prost.-vrem. = konst. × $\underbrace{\text{materija}}_{\text{svi oblici energije koji posjeduju masu}}$

"Ricci = energija"

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \varkappa\, T_{\mu\nu}, \qquad \varkappa = -\frac{8\pi G_N}{c^4}$$

Einsteinova j-ba

OTR ne govori o konstanti vezanja

prepoznaje se za male zakrivljenosti – slaba gravitac. polja

skup 10 j-bi ⇒

Riemann = Ricci + Weyl (mjeri plimna naprezanja)

20 komp = 10 + 10

EINSTEINOVE JEDNADŽBE

- Einstein equations : principle of stationary action applied to GR

cosmological constant

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (R - \Lambda) = \frac{8\pi G}{c^4} T_{\mu\nu}$$

curvature tensor (Ricci tensor): built from $g_{\mu\nu}$ and its derivatives

scalar curvature $R \equiv g^{\mu\nu} R_{\mu\nu}$

energy-momentum tensor : energy content of the universe

geometry

- FLRW metric: assuming space is filled with an homogenous fluid of pressure P and density ρ

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}\left(R-\Lambda\right)=\frac{8\pi G}{c^4}T_{\mu\nu}$$

μ=ν=0

$$3\left(\frac{\dot{a}^2+k}{a^2}\right)-\Lambda=8\pi G\rho$$

energy content

space-time geometry

- k=+1 closed
- k=0 flat
- k=-1 open

with $T_{\mu\nu}=\begin{bmatrix}\rho g_{00} & 0 & 0 & 0\\ 0 & -Pg_{11} & 0 & 0\\ 0 & 0 & -Pg_{22} & 0\\ 0 & 0 & 0 & -Pg_{33}\end{bmatrix}$

Einsteinova jednadžba vrijedi kovarijantno - sugibajuće koordinate transformiramo u koordinate slobodnog pada, gdje su nam poznati izrazi za tenzor energije-impulsa (Vježba 3.1 & 3.2)

$$T^{\mu\nu}(x) = \frac{\partial x^\mu}{\partial \xi^\alpha}\frac{\partial x^\nu}{\partial \xi^\beta} T^{\alpha\beta}(\xi=0) \;; \quad x = x(\xi=0)$$

$$T^{00} = \rho \;, \quad T^{ii} = p = -p\,\eta^{ii}$$

$$(i = r, \vartheta, \varphi)$$

$$T^{\mu\nu} = U^\mu U^\nu \rho + (U^\mu U^\nu - g^{\mu\nu})\, p$$

$$U^\mu = \frac{\partial x^\mu}{\partial \xi^0}$$ 4-brzina $\left\{ \begin{array}{l} x\text{-a u odn. na } \xi \\ \text{č. u sust. } x \;(=0 \text{ u sust. } \xi) \end{array} \right.$

Lijeva strana Einsteinove jedn.

- Friedmannove j-be
  - kao specifikacija Einsteinovih BEZ kozmol. člana, na IDEALIZIRANI SVEMIR s Robertson-Walkerovom metrikom

    kojom izražavamo lijevu stranu Einst. j-be

"LHS" – Einsteinov tenzor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

$$G^1_1 \equiv R^1_1 - \frac{1}{2} R = -\frac{1}{c^2}\left(2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + k c^2}{S^2}\right) = G^2_2 = G^3_3$$

$$G^0_0 \equiv R^0_0 - \frac{1}{2} R = -\frac{3}{c^2}\left(\frac{\dot{S}^2 + k c^2}{S^2}\right)$$

& desna strana jednadžbe

"RHS" – desna strana Einsteinove j-be
određena tenzorom energije-impulsa, $T_{\mu\nu}$

$$-\frac{8\pi G}{c^4} T_{\mu\nu} \qquad \begin{cases} T^0_0 = \varepsilon \\ \underbrace{T^1_1 = T^2_2 = T^3_3}_{\text{izotropnost!}} = -p \end{cases}$$

$$2\frac{\ddot{S}}{S} + \frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{c^2} T^i_i \quad ; \; i = 1, 2, 3 \qquad (1)$$

$$\frac{\dot{S}^2 + kc^2}{S^2} = \frac{8\pi G}{3c^2} T^0_0 \qquad (2)$$

$$T^\mu_\nu = T^\mu_\nu\Big|_{\text{tvari}} + T^\mu_\nu\Big|_{\text{zračenja}}$$

$\xrightarrow{\text{URL}}$ diag$(\varepsilon, -\varepsilon/3, -\varepsilon/3, -\varepsilon/3)$

$T^0_0 = \varepsilon$

VEZA DVIJU JEDNADŽBI

$(1)-(2) \Rightarrow$

$$2\frac{\ddot{S}}{S} = -\frac{8\pi}{3c^2}G\left(\rho c^2 + 3p\right)$$

$\frac{d}{dt}\left[(1)\times S^2\right] \Rightarrow$

$$2\dot{S}\ddot{S} = \frac{8\pi}{3}G\left(\dot{\rho}S^2 + 2\rho S\dot{S}\right)$$

$$\boxed{\dot{\rho}c^2 S + 3\left(\rho c^2 + p\right)\dot{S} = 0}$$

DZ 3.1:
Slično pokazati

$$\frac{d}{dS}\left(\rho c^2 S^3\right) + 3pS^2 = 0$$

Uvođenje jednadžbe stanja - uz 2 nezavisne (za 3 nepoznanice)

- Friedmannova j.
- J. Fluida
- J. ubrzanja

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}\varepsilon - \frac{\kappa c^2}{R_0^2 a}$$

$$\dot{\varepsilon} + 3\frac{\dot{a}}{a}(\varepsilon + P) = 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\varepsilon + 3P)$$

$P = w\varepsilon$ kozmološki primjerena linearna relacija

Conservation of the energy-momentum tensor $T_{\mu\nu}$: for each component described as a fluid of pressure P and density ρ

$$\frac{\partial \rho}{\partial t} + 3H(P+\rho) = 0$$

further assumption: $P = w\ \rho$ equation of state

radiation or relativistic matter (w=1/3):

$$P_r = \frac{\rho_r}{3} \Rightarrow \frac{\partial \rho_r}{\partial t} = -4H\rho_r \Rightarrow \rho_r(t) = \rho_r^0 \left(\frac{a_0}{a(t)}\right)^4$$

non-relativistic matter (w=0):

$$P_m = 0 \Rightarrow \frac{\partial \rho_m}{\partial t} = -3H\rho_m \Rightarrow \rho_m(t) = \rho_m^0 \left(\frac{a_0}{a(t)}\right)^3$$

cosmological constant ⇔ fluid with w=-1: $\rho_\Lambda(t) = \rho_\Lambda^0 = \frac{\Lambda}{8\pi G}$

Uvođenje parametra gustoće

- parametar gustoće

$$\Omega = \rho / \rho_c \ ; \ \rho_c = \frac{3H^2}{8\pi G}$$

$$\rho_{c,0} = 2.76 \cdot 10^{11} \, h^2 \, M_\odot \, Mpc^{-1}$$

$$-kc^2 = \dot{S}^2 - \frac{8\pi G}{3} \rho S^2 - \frac{\lambda}{3} S^2$$

$$= S^2 H^2 \left[1 - \frac{\rho_m + \rho_r + \rho_\lambda}{\rho_c} \right] \quad \& \quad \Omega_{m,r,\lambda} = \frac{\rho_{m,r,\Lambda}}{\rho_c}$$

$$-kc^2 = S^2 H^2 \left[1 - (\Omega_m + \Omega_r + \Omega_\lambda) \right]$$