FIZIKA ELEMENTARNIH ČESTICA
VJEŽBE

PRIRODNI SUSTAV JEDINICA

- kada se bavimo elementarnim česticama neprirodno je koristiti S.I. sustav jedinica u kojem je $m_e = 9{,}1 \cdot 10^{-31}\,kg$ ili $c = 299\,792\,558\ m/s$.
- kako bi izbjegli jako velike i jako male brojeve pribjegavamo novom sustavu jedinica;

Prirodni sustav jedinica

$c = \hbar = 1$ odakle se dobije i $k_B = 1$

c i $\hbar$ su u ovom sustavu bezdimenzionalni i poprimaju jedinicu

u S.I.

$\hbar = 1{,}055 \cdot 10^{-34}\,J\,s$

$c = 2{,}998 \cdot 10^{8}\,m\,s^{-1}$

- uz to potrebna nam je još jedna jedinica, kako bi izgradili upotrebljiv sustav jedinica
- u fizici elementarnih čestica ta jedinica je elektronvolt (eV) [jedinica energije]
- $eV = 1{,}602 \cdot 10^{-19}\,J$

Primjer 1.

1. jedinica duljine u PSJ

u S.I.

$$[L] \sim \frac{c\hbar}{eV} = \frac{2{,}998 \cdot 10^{8}\,m\,s^{-1} \cdot 1{,}055 \cdot 10^{-34}\,J\,s}{1{,}602 \cdot 10^{-19}\,J} = 1{,}974 \cdot 10^{-7}\,m$$

u PSJ

$$\frac{c\hbar}{eV} = 1\,eV^{-1} = 1{,}974 \cdot 10^{-7}\,m$$

$1\,eV^{-1}$ duljine u PSJ odgovara $1{,}974 \cdot 10^{-7}\,m$ u S.I.

2. jedinica vremena u PSJ

$$[T] \sim \frac{\hbar}{eV} = \frac{1{,}055 \cdot 10^{-34}\,J\,s}{1{,}602 \cdot 10^{-19}\,J} = 6{,}586 \cdot 10^{-16}\,s = 1\,eV^{-1}$$

3. jedinica mase u PSJ

- iz Einsteinove relacije

$$E = mc^2 \Rightarrow m = \frac{E}{c^2}$$

slijedi

$$[M] \sim \frac{eV}{c^2} = \frac{1{,}602 \cdot 10^{-19}\,J}{(2{,}998 \cdot 10^{8})^2\,m^2 s^{-2}} = \frac{1{,}602 \cdot 10^{-19}\,kg\,m^2 s^{-2}}{(2{,}998 \cdot 10^{8})^2\,m^2 s^{-2}} = 1{,}782 \cdot 10^{-36}\,kg = 1\,eV$$

Heaviside-Lorentzov sustav jedinica

- jednadžbe klasične elektrodinamike se mogu pojednostavniti uvođenjem Heaviside-Lorentzovih jedinica.
- vrijednost naboja elektrona je definirana silom između dva elektrona na udaljenosti r:

S.I.

$$F = \frac{e^2}{4\pi\varepsilon_0 r^2}$$

Heaviside-Lorentz ($\varepsilon_0 = 1$) – permitivnost vakuuma

$$F = \frac{e^2}{4\pi r^2}$$

- apsorbirali smo ε_0 u definiciju naboja
- Kombinirajući prirodni sustav jedinica u kojem je $c=1 \Rightarrow \frac{1}{\mu_0 \varepsilon_0} = c = 1 \Rightarrow \mu_0 = 1$ - permeabilnost vakuuma
- Kombinirani sustav jedinica:

$$c = \hbar = \mu_0 = \varepsilon_0 = 1$$

- korisno je uvesti bezdimenzionalnu konstantu fine strukture: (ista u svakom sustavu jedinica)

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137}$$

- naboj elektrona se prema tome može izraziti kao $\boxed{e = \sqrt{4\pi\varepsilon_0 \hbar c \alpha}}$
- u $c = \hbar = \mu_0 = \varepsilon_0 = 1$ sustavu $e = \sqrt{4\pi\alpha} = 0{,}303$ jediničnih naboja
- jedinični naboj u našem sustavu je $1_{gn} = \frac{e}{\sqrt{4\pi\alpha}} = 5{,}291 \cdot 10^{-19}\,C$

Primjer 2.

Masa Higgsovog bozona u prirodnom sustavu jedinica je $m_H = 125\,GeV$! Izrazite masu u S.I. sustavu

$$m_H = 125 \cdot 10^9 \cdot 1{,}782 \cdot 10^{-36}\,kg = 2{,}2275 \cdot 10^{-25}\,kg$$

SPECIJALNA TEORIJA RELATIVNOSTI - RELATIVISTIČKA KINEMATIKA

- Kontravarijantni vektor:
$$x^\mu = (t, \vec{x}) = (t, x^1, x^2, x^3)$$
- Kovarijantni vektor
$$x_\mu = (t, -\vec{x}) = (t, -x^1, -x^2, -x^3)$$

Napomena za predavanje: Kovarijantni vektori se transformiraju kao $\frac{\partial}{\partial x^\mu}$

- Veza između kontravarijantnih i kovarijantnih vektora je dana izrazom:
$$x_\mu = g_{\mu\nu} x^\nu \quad \text{gdje je} \quad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \text{metrika ravnog prostora}$$

- U fizici elementarnih čestica voljeli bi fizikalne veličine koje mjerimo ili računamo izraziti kao Lorentz invarijantne veličine: (udarni presjeci, širine raspada itd...)
- Najjednostavniji primjer je Lorentz invarijantni prostorno-vremenski interval
$$x_\mu x^\mu = t^2 - x^2 - y^2 - z^2$$
- Sami vektori (x^μ, A^μ, p^μ ...) se na Lorentzove transformacije transformiraju kao:
$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$
- $\Lambda^\mu{}_\nu$ su elementi matrice $\hat{\Lambda}$ koja predstavlja Lorentzovu transformaciju. $\hat{\Lambda}(\beta)$ ovisi o β (brzini između dva sustava)

Primjer 1.

Boost u x-smjeru

$$\Lambda^\mu{}_\nu(\beta) = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad x' = \begin{pmatrix} t' \\ x_1' \\ x_2' \\ x_3' \end{pmatrix} = \Lambda^\mu{}_\nu(\beta) x^\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma(t - \beta x_1) \\ \gamma(x_1 - \beta t) \\ x_2 \\ x_3 \end{pmatrix}$$

- Lorentz invarijantnost skalarnog produkta vodi na relacije ortogonalnosti Lorentzovih transformacija
$$x_\mu x^\mu \overset{!}{=} x'_\mu x'^\mu = \Lambda_\mu{}^\beta x_\beta \Lambda^\mu{}_\sigma x^\sigma \Rightarrow \Lambda_\mu{}^\beta \Lambda^\mu{}_\sigma = \mathbb{1} = \delta^\beta{}_\sigma$$

Primjer 2.

Za boost u z-smjeru pokažite da je $\Lambda^\mu{}_\nu(-\beta)$ transformacija inverzna transformaciji $\Lambda^\mu{}_\nu(\beta)$!

$$\Lambda^\mu{}_\nu(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$\Lambda^\mu{}_\nu(\beta)\Lambda^\nu{}_\sigma(-\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & \gamma^2\beta - \gamma^2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma^2\beta + \gamma^2\beta & 0 & 0 & \gamma^2(1-\beta^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Energija i moment

$$E = \gamma m \qquad \vec{p} = \gamma m \vec{\beta} \quad \Rightarrow \quad |\vec{p}| = E\beta$$

Primjer 3.

Pokažite da energija i impuls čine komponente četverovektora! Koristiti dilataciju vremena $\frac{d\tau}{dt} = \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\gamma}$

τ - vlastito vrijeme

$$E = \gamma m = \gamma m \frac{dt}{dt} = \gamma m \frac{dt}{d\tau}\frac{d\tau}{dt} = m\frac{dt}{d\tau}$$ - transformira se kao vremenska komponenta

$$\vec{p} = \gamma m \vec{\beta} = \gamma m \frac{d\vec{x}}{dt} = \gamma m \frac{d\vec{x}}{d\tau}\frac{d\tau}{dt} = m\frac{d\vec{x}}{d\tau}$$ - transformira se kao prostorna komponenta

- uvodimo četverovektor $p^\mu = (E, \vec{p})$

$$p_\mu p^\mu = E^2 - \vec{p}^{\,2} = m^2$$

- u fizici elementarnih čestica najčešće proučavamo sudare i raspade
- najčešći tip sudara je 2 u 2.

$$1+2 \to 3+4$$

$p_i^\mu = p_1^\mu + p_2^\mu$ inicijal

$p_f^\mu = p_3^\mu + p_4^\mu$ final

- zakon očuvanja za energiju i impuls

$$p_i = p_f$$

- u proučavanju ovih sudara vrlo korisne su Lorentz invarijante: Mandelstamove varijable

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t = (p_1-p_3)^2 = (p_2-p_4)^2$$
$$u = (p_1-p_4)^2 = (p_2-p_3)^2$$

- u sustavu centra mase (SCM):

$$p_1 = (E_1, \vec{p}) \;;\; p_2 = (E_2, -\vec{p})$$

$$\Rightarrow \quad s = (p_1+p_2)^2 = (E_1+E_2)^2 - (\vec{p}-\vec{p})^2 = (E_1+E_2)^2$$

- slijedi da je $\sqrt{s}$ ukupna dostupna energija u sustavu centra mase
- ta energija je dostupna za stvaranje novih čestica ($\sqrt{s}$ daje prag za stvaranje novih čestica)

Primjer 4.

LAB: $\overset{A}{\underset{p_1}{\rightarrow}}\ \overset{B}{\underset{p_2}{\leftarrow}}$, $\underset{p_3}{\leftarrow}$ c; $\sqrt{s} = E_{A\,scm} + E_{B\,scm} = m_c$

SCM: $\overset{A}{\underset{p_A}{\rightarrow}}\ \overset{B}{\underset{-p_A}{\leftarrow}}$, c; $E_{c\,scm} = E_A + E_B \Rightarrow \boxed{m_c \le E_A + E_B}$

Collider (sudarivač)

$$p_1^\mu = (E_1, \vec{p}_1) \;;\; p_2^\mu = (E_2, \vec{p}_2)$$ - podsjeimo općenito

$$s = (p_1+p_2)^2 = (E_1+E_2)^2 - (\vec{p}_1+\vec{p}_2)^2 = \underbrace{E_1^2}_{} + 2E_1E_2 + E_2^2 - \vec{p}_1^{\,2} - 2\vec{p}_1\cdot\vec{p}_2 - \vec{p}_2^{\,2} = m_1^2 + m_2^2 + 2E_1E_2 - 2|\vec{p}_1||\vec{p}_2|\cos\theta$$

- za sudarivač kut je $\theta = \pi$

$$\Rightarrow \quad s = m_1^2 + m_2^2 + 2E_1E_2(1+\beta_1\beta_2)$$

- u ultrarelativističkom limesu (URL)

$$E_1, E_2 \gg m_1, m_2 \;;\; \beta_1, \beta_2 \approx 1 \quad \Rightarrow \quad \boxed{s = 4E_1E_2} \quad \text{za } E = E_1 = E_2 \Rightarrow \sqrt{s} = 2E$$

Primjer 5.

Fixed target

$p_1^\mu = (E_1, \vec p_1)$

$p_2^\mu = (m_2, \vec 0)$

$s = (p_1+p_2)^2 = (E_1+m_2)^2 - (\vec p_1 + \vec 0)^2 = \overbrace{E_1^2 + 2E_1 m_2 + m_2^2 - \vec p_1^{\,2}}^{m_1} = m_1^2 + m_2^2 + 2E_1 m_2$

- URL:

$E_1 \gg m_1, m_2 \Rightarrow s = 2E_1 m_2 \Rightarrow \sqrt{s} = \sqrt{2E_1 m_2}$

Primjer 6.

- zašto je sudarivač bolji od fixed target eksperimenta?
- LHC do sada radio na 4 TeV po zraci
- sudarano protone

$E_1 = E_2 = 4\,\text{TeV} \Rightarrow \sqrt{s} = 8\,\text{TeV}$ dostupno za proizvodnju nove čestice

- kada bi jedan proton mirovao za istu dostupnu energiju trebali bi zraku energije

$\sqrt{s} = 8\,\text{TeV};\ m_p = 938\,\text{MeV} = 9{,}38\cdot10^{-4}\,\text{TeV}$

$E_p = ?$ $\quad 8\,\text{TeV} = \sqrt{2m_p E_p} \Rightarrow E_p = 3{,}4\cdot10^{4}\,\text{TeV}$, jedna zraka bi trebala biti 10 000 puta snažnija

- LEP (nije više u funkciji)
- sudarao elektrone i pozitrone

$E_{e^-} = E_{e^+} = 45{,}5\,\text{GeV}$ - za stvaranje Z bozona

$\sqrt{s} = 91\,\text{GeV} = m_{Z^0}$

- ako $\vec p_{e^+} = 0$ trebali bi $E_{e^-} = ?$

$\sqrt{s} = 91\,\text{GeV}$

$m_{e^+} = 0{,}5\,\text{MeV} = 5\cdot10^{-4}\,\text{GeV}$

$E_{e^-} = \frac{91^2}{2m_{e^+}} = 8{,}3\cdot10^{6}\,\text{GeV}$

Primjer 7.

- raspad Z^0 bozona na mion i antimion

$Z^0 \to \mu^+\mu^-$

- odrediti duljinu preleta miona

$m_{Z^0} = 91\,\text{GeV}$

$m_\mu = 0{,}1\,\text{GeV}$

$\tau_\mu = 2{,}2\cdot10^{-6}\,\text{s}$

$p_{Z^0}^\mu = (m_Z, \vec 0)$

$p^\mu_{\mu^+} = (E_+, \vec p_+)$ $\Rightarrow$ $E_+ = E_- = \frac{m_Z}{2}$, $\vec p_+ = -\vec p_-$ $\Rightarrow$ $E_\mu^2 = |\vec p|^2 + m_\mu^2 \Rightarrow |\vec p|^2 = \frac{m_Z^2}{4} - m_\mu^2$

$p^\mu_{\mu^-} = (E_-, \vec p_-)$

$|\vec p| = \sqrt{\frac{m_Z^2}{4} - m_\mu^2} = \beta\cdot\cancel{\frac{E_Z}{2}} = \beta\cdot\frac{m_Z}{2}$

$\Rightarrow \beta = \frac{2}{m_Z}\sqrt{\frac{m_Z^2}{4} - m_\mu^2} = \sqrt{1 - \frac{4m_\mu^2}{m_Z^2}}$

$= 0{,}9999976$

$\Delta t = \tau_\mu \cdot \gamma$

$\Rightarrow L = \Delta t\cdot\beta c = \tau_\mu\cdot\frac{\beta c}{\sqrt{1-\beta^2}} = 301\,\text{km}$

Primjer 8.

- raspad Higgsovog bozona na τ^+ i τ^- leptone

  $h \to \tau^+ \tau^-$

- odrediti duljinu proleta τ bozona

$m_h = 125 \text{ GeV}$

$m_{\tau^+} = m_{\tau^-} = 1,7 \text{ GeV}$

$t_\tau = 2,9 \cdot 10^{-13} \text{ s}$

$p^\mu_h = (m_h, \vec{0})$

$p^\mu_{\tau^+} = (E_+, \vec{p}_+)$ $\Rightarrow$

$p^\mu_{\tau^-} = (E_-, \vec{p}_-)$

ZOE, ZOI

$E_+ = E_- = \frac{m_h}{2}$

$\vec{p}_+ = -\vec{p}_-$ $\Rightarrow$

$E_\tau^2 = |\vec{p}|^2 + m_\tau^2$

$\Rightarrow |\vec{p}|^2 = \frac{m_h^2}{4} - m_\tau^2$

$|\vec{p}| = \sqrt{\frac{m_h^2}{4} - m_\tau^2} = \beta E_\tau = \beta \cdot \frac{m_h}{2}$

$\Rightarrow \beta = \frac{2}{m_h}\sqrt{\frac{m_h^2}{4} - m_\tau^2} = \sqrt{1 - \frac{4m_\tau^2}{m_h^2}}$

$= 0,99963$

Dilatacija vremena

$\Delta t = t_\tau \cdot \gamma$

$\Rightarrow L = \Delta t \cdot \beta \cdot c = t_\tau \cdot \frac{\beta c}{\sqrt{1-\beta^2}} = \underline{3,2 \cdot 10^{-3} \text{ m}}$

PARITET

- diskretna transformacija
- ostale su (C, P, T) — P: paritet

- P: - operacija prostorne inverzije koordinata

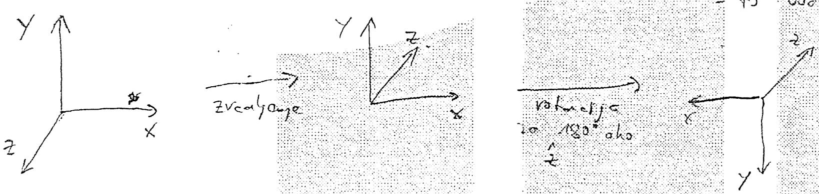
$$P: \vec{r} \longrightarrow \vec{r}\,' = -\vec{r} \qquad \text{tj.} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

$$P: A^{\mu} = (A^0, \vec{A}) \longrightarrow A^{\mu'} = (A^0, -\vec{A})$$

- transformacija pariteta jednaka je zrcaljenju u nekoj ravnini + rotacija u toj ravnini

Paritet = zrcaljenje + rotacija

↳ rotacija ne mijenja fiziku (teorija je invarijantna na rotacije u prostoru)
 - to vodi na sačuvanje angularnog momenta



- [illegible] vektora

$\vec{r} = -\vec{r}$

$\vec{p} = -\vec{p}$

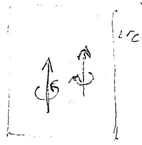
postoje i pseudovektori:

- npr. $\vec{L} = \vec{r} \times \vec{p} \xrightarrow{P} \vec{L}' = \vec{r}\,' \times \vec{p}\,' = \vec{r} \times \vec{p} = \vec{L}$ (angularni moment je pseudovektor)

	P	zrcaljenje u xy ravnini
r_x, p_x	$-$	$+$
r_y, p_y	$-$	$+$
r_z, p_z	$-$	$-$
L_x	$+$	$-$
L_y	$+$	$-$
L_z	$+$	$+$

ZRCALJENJE:

- komponente angularnog momenta (spina) okomite na ravninu zrcaljenja ostaju iste, a komponente u ravnini mijenjaju smjer



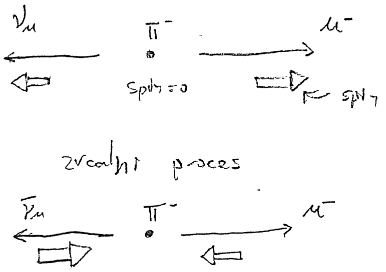
- Kada interakcija čuva paritet?
- Ako je vjerojatnost prijelaza $|1\rangle \to |2\rangle$ jednaka vjerojatnosti prijelaza $P|1\rangle \to P|2\rangle$

tj. $\langle 2|O|1\rangle = \langle 2|P^{-1}OP|1\rangle$ (O je operator prijelaza)

P - operator pariteta $\Rightarrow O = P^{-1}OP \Rightarrow [O,P]=0$ tj. budući da je $O \sim \exp(iHt)$
$[H,P]=0$

Primjer

- raspad $\pi^- \to \mu^- + \bar{\nu}_\mu$ $\pi^- = d\bar{u}$
- definirano helicitet: helicitet je projekcija spina na impuls (smjer gibanja)

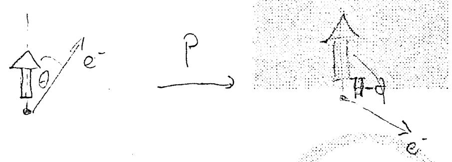


- da bi interakcija kojom se raspada pion π^- bila invarijantna na P oba procesa moraju biti jednako vjerojatna
- eksperimentalno zrcalni proces nije opažen!
- helicitet μ^- je uvijek $+1/2$; $\bar{\nu}_\mu$ uvijek $+1/2$
- za ovaj proces je odgovorna slaba sila (narušava paritet)
- elektromagnetska i jaka sila čuvaju paritet

Primjer

- raspad miona

$\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$



$\theta = \angle(\vec{S}_\mu, \vec{p}_{e^-})$

- ako je paritet sačuvan

P očuvan $\Rightarrow$ $N_{e^-}(\theta) = N_{e^-}(\pi - \theta)$ - vjerojatnosti iste
- eksperimentalno ne vrijedi

$N(\theta) = 1 + a\cos\theta$; $a = \frac{1}{3} \Rightarrow N(\theta) \neq N(\pi - \theta)$

Uvodimo operaciju pariteta na stanje s dvije čestice

$P|AB\rangle = (-1)^L P|A\rangle P|B\rangle$; L - orbitalni kvantni broj

zašto? $\Rightarrow$ $P\psi_L^m(r,\theta,\phi) = (-1)^L \psi_L^m(r,\theta,\phi)$

- čestice $|A\rangle$ i $|B\rangle$ interagiraju centralnom silom. Njihovo međusobno gibanje dano je valnom jednadžbom $\psi_L^m(r,\theta,\phi) = R(r)\, Y_L^m(\theta,\phi)$
  - kuglaste funkcije (sferni harmonici)

- pod operacijom pariteta imamo $\boxed{\begin{array}{l} r \to r \\ \theta \to \pi - \theta \\ \phi \to \pi + \phi \end{array}}$

Zadatak

Za proces $\Delta^{++} \to p\pi^+$ odgovorna je jaka nuklearna sila (čuva paritet)

Odredite angularni moment između čestica p i π^+! ($L=?$)

$\Delta^{++} = uuu \qquad J^P(\Delta^{++}) = \frac{3}{2}^+$

$p = uud \qquad J^P(p) = \frac{1}{2}^+$

$\pi^+ = u\bar{d} \qquad J^P(\pi^+) = 0^-$

– gledamo paritet

$$P: \underset{\Delta^{++}}{+1} = \underset{p}{(+1)}\,\underset{\pi^+}{(-1)}\,\underset{\text{relativno gibanje}}{(-1)^L} \Rightarrow L = \text{neparan}$$

– s druge strane (kako možemo iskombinirati L i $j_p = \frac{1}{2}$ u $j_{\Delta^{++}} = \frac{3}{2}$)

$$|j_1 - j_2| \le j \le |j_1 + j_2| \Rightarrow \left|\tfrac{1}{2} - L\right| \le \tfrac{3}{2} \le \left|\tfrac{1}{2} + L\right| \Rightarrow L = 1 \text{ ili } 2$$

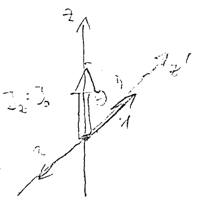
– slijedi da je $\underline{L=1}$ //

Zadatak

Gledamo proces $0 \to 1+2$ (recimo iz prošlog zadatka $\Delta^{++} \to p\pi^+$), takav da $J_1 = \frac{1}{2}$, $J_2 = 0$, $J_0 = \frac{3}{2}$

– treba odrediti ovisnost o kutu θ da detektiramo česticu 1

– uzimamo koordinatni sustav takav da je $J_{0z} = max$



– raspisujemo stanje $|J, J_z\rangle$ po stanjima $|J, J_{z'}\rangle$

$$|J, J_z\rangle = \sum_{J_{z'}} d^J_{J_{z'}J_z}(\theta)\,|J, J_{z'}\rangle$$

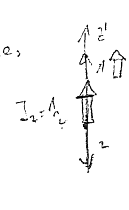
$$|J=\tfrac{3}{2}, J_z=\tfrac{3}{2}\rangle = d_1\,|J=\tfrac{3}{2}, J_{z'}=\tfrac{3}{2}\rangle + d_2\,|J=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2}\rangle + d_3\,|J=\tfrac{3}{2}, J_{z'}=-\tfrac{1}{2}\rangle + d_4\,|J=\tfrac{3}{2}, J_{z'}=-\tfrac{3}{2}\rangle$$

– vjerojatnost raspada! Čestice 1, 2 na z' osi mogu doći samo $|J=\frac{3}{2}, J_z=\frac{1}{2}\rangle$ ili $|J=\frac{3}{2}, J_z=-\frac{1}{2}\rangle$ jer $J(p)=\frac{1}{2}$, Relativno gibanje nema komponentu [illegible] z' $L_{z'}=0$!

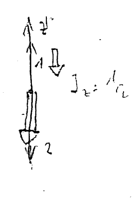
$$W(\theta) = \left|{}_f\langle J=\tfrac{1}{2}, J_{z'}=\tfrac{1}{2}|\,O\,d_2\,|J=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2}\rangle_i\right|^2 + \left|{}_f\langle J=\tfrac{1}{2}, J_{z'}=-\tfrac{1}{2}|\,O\,d_3\,|J=\tfrac{3}{2}, J_{z'}=-\tfrac{1}{2}\rangle_i\right|^2$$

$$= \left|d^{3/2}_{1/2\,3/2}(\theta)\,A_{1/2}\right|^2 + \left|d^{3/2}_{-1/2\,3/2}(\theta)\,A_{-1/2}\right|^2$$

$A_{1/2}$ je amplituda za proces



, a $A_{-1/2}$ za proces



Napomena:

Wignerove d funkcije! Operator rotacije $R(\alpha\beta\gamma) = e^{-i\alpha J_z}\,e^{-i\beta J_y}\,e^{-i\gamma J_z}$! $\alpha\beta\gamma$ – Eulerovi kutevi

Wignerova D-matrica $D^j_{m'm}(\alpha,\beta,\gamma) = \langle jm'|R(\alpha,\beta,\gamma)|jm\rangle = e^{-im'\alpha}\,d^j_{m'm}(\beta)\,e^{-im\gamma}$

$$d^j_{m'm}(\beta) = \langle jm'|e^{-i\beta J_y}|jm\rangle$$

IZOSPIN

– masa protona i neutrona približno ista

$m_N = \cancel{939{,}5} \simeq m_p = 938{,}3\ MeV$
939,5

– jaka nuklearna sila ih tretira kao iste (razlika je elektromagnetska)

$\begin{pmatrix} p \\ n \end{pmatrix}$ – uvodimo SU(2) dublet
– fizika je simetrična na izmjenu $p \leftrightarrow n$; na rotaciju u izospinskom prostoru

$|p\rangle = |I = 1/2, I_3 = 1/2\rangle$ p = uud
$|n\rangle = |I = 1/2, I_3 = -1/2\rangle$ n = udd

$m_{\pi^-} = m_{\pi^+} = 139{,}6\ MeV \simeq m_{\pi^0} = 135{,}0\ MeV$
– njih smještamo u reprezentaciju izospina 1

$\begin{pmatrix} \pi^+ \\ \pi^0 \\ \pi^- \end{pmatrix}$
$|\pi^+\rangle = |I = 1, I_3 = 1\rangle$
$|\pi^0\rangle = |I = 1, I_3 = 0\rangle$
$|\pi^-\rangle = |I = 1, I_3 = -1\rangle$

napomena pioni
$\pi^+ = u\bar{d}$
$\pi^0 = u\bar{u}, d\bar{d}$
$\pi^- = d\bar{u}$

– nadalje sigma barioni i kaoni

$\begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix}$
$\Sigma^+ = uus$
$\Sigma^0 = uds$
$\Sigma^- = dds$

$\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}$ $K^+ = u\bar{s}$, $K^0 = d\bar{s}$

$\begin{pmatrix} \bar{K}^0 \\ K^- \end{pmatrix}$ $\bar{K}^0 = \bar{d}s$, $K^- = \bar{u}s$

$I_3 = \frac{1}{2}\left[(n_u - n_{\bar{u}}) - (n_d - n_{\bar{d}})\right]$

– jaka sila ne mijenja izospin
– čak ako i stvara $q\bar{q}$ par izospin ostaje isti

Primjeri

Pion-nukleon raspršenje

npr. $\pi^+ + n \to \pi^0 + p$

$|\pi^+ n\rangle = |1,1\rangle|\tfrac12,-\tfrac12\rangle = \sqrt{\tfrac13}\,|\tfrac32,\tfrac12\rangle + \sqrt{\tfrac23}\,|\tfrac12,\tfrac12\rangle$

$|\pi^0 p\rangle = |1,0\rangle|\tfrac12,\tfrac12\rangle = \sqrt{\tfrac23}\,|\tfrac32,\tfrac12\rangle - \sqrt{\tfrac13}\,|\tfrac12,\tfrac12\rangle$

Prijelaz je opisan izrazom

$\langle \pi^0 p | O_s | \pi^+ n\rangle = \sqrt{\tfrac23}\sqrt{\tfrac13}\langle \tfrac32 \tfrac12 | O_s | \tfrac32 \tfrac12\rangle - \sqrt{\tfrac23}\sqrt{\tfrac13}\langle \tfrac12,\tfrac12 | O_s | \tfrac12,\tfrac12\rangle = \frac{\sqrt2}{3} A_{3/2} - \frac{\sqrt2}{3} A_{1/2}$

$\longrightarrow$ O_s operator prijelaza (jaka sila – ne mijenja izospin)

Zadatak

Pogledajte dva procesa:

A: $K^- p \to \Sigma^0 \pi^0$

B: $K^- p \to \Sigma^+ \pi^-$

Odredite omjer udarnih presjeka za ova dva procesa?

$\frac{\sigma_A}{\sigma_B} = ?$

a) $|K^- p\rangle = |\tfrac12, -\tfrac12\rangle |\tfrac12, +\tfrac12\rangle = CG_1 |1,0\rangle + CG_2|0,0\rangle = \sqrt{\tfrac12}\,|1,0\rangle - \sqrt{\tfrac12}\,|0,0\rangle$

$|\Sigma^0\pi^0\rangle = |1,0\rangle|1,0\rangle = \sqrt{\tfrac23}\,|2,0\rangle + 0\,|1,0\rangle - \sqrt{\tfrac13}\,|0,0\rangle$

$$A_A \sim \langle f|O_s|i\rangle = \sqrt{\tfrac23}\sqrt{\tfrac12}\underbrace{\langle 2,0|O_s|1,0\rangle}_{0 \text{ jer } O_s \text{ ne mijenja izospin}} + \sqrt{\tfrac23}\sqrt{\tfrac12}\underbrace{\langle 2,0|O_s|0,0\rangle}_{0} - \sqrt{\tfrac13}\sqrt{\tfrac12}\underbrace{\langle 0,0|O_s|1,0\rangle}_{0} + \sqrt{\tfrac13}\sqrt{\tfrac12}\underbrace{\langle 0,0|O_s|0,0\rangle}_{A_0}$$

$\Rightarrow A_A \sim \sqrt{\tfrac16}\, A_0$

b) $|\Sigma^+\pi^-\rangle = |1,1\rangle|1,-1\rangle = \sqrt{\tfrac16}\,|2,0\rangle + \sqrt{\tfrac12}\,|1,0\rangle - \sqrt{\tfrac13}\,|0,0\rangle$

$A_B \sim \frac12 \underbrace{\langle 1,0|O_s|1,0\rangle}_{A_1} - \sqrt{\tfrac16}\underbrace{\langle 0,0|O_s|0,0\rangle}_{A_0}$

$A_B \sim \frac12 A_1 - \sqrt{\tfrac16}\, A_0$

omjer udarnih presjeka

$$\frac{\sigma_A}{\sigma_B} = \frac{|A_A|^2}{|A_B|^2} = \frac{|\sqrt{\tfrac16}A_0|^2}{|\tfrac12 A_1 - \sqrt{\tfrac16}A_0|^2}$$

za $A_0 \gg A_1 \Rightarrow \frac{\sigma_A}{\sigma_B} = 1$

za $A_1 \gg A_0 \Rightarrow \frac{\sigma_A}{\sigma_B} = 0$

Zadatak

Izrazite amplitudu procesa $\pi^0 n \to \pi^- p$ preko amplituda za procese $\pi^+ p \to \pi^+ p$; $\pi^- p \to \pi^- p$

a) $\pi^+ p \to \pi^+ p$

$|\pi^+ p\rangle = |1,1\rangle|\tfrac12,\tfrac12\rangle = |\tfrac32, \tfrac32\rangle$

$S_{\pi^+ p} = \langle \pi^+ p|O_s|\pi^+ p\rangle = \langle \tfrac32,\tfrac32|O_s|\tfrac32,\tfrac32\rangle = A_{3/2}$

b) $\pi^- p \to \pi^- p$

$|\pi^- p\rangle = \sqrt{\tfrac13}\,|\tfrac32,-\tfrac12\rangle - \sqrt{\tfrac23}\,|\tfrac12,-\tfrac12\rangle$

$S_{\pi^- p} = \langle\pi^- p|O_s|\pi^- p\rangle = \tfrac13\langle \tfrac32,-\tfrac12|O_s|\tfrac32,-\tfrac12\rangle + \tfrac23\langle \tfrac12,-\tfrac12|O_s|\tfrac12,-\tfrac12\rangle$

$= \tfrac13 A_{3/2} + \tfrac23 A_{1/2}$

c) $\pi^0 n \to \pi^- p$

$|\pi^0 n\rangle = \sqrt{\tfrac23}\,|\tfrac32,-\tfrac12\rangle + \sqrt{\tfrac13}\,|\tfrac12,-\tfrac12\rangle$

$S = \langle \pi^0 n|H|\pi^- p\rangle = \frac{\sqrt2}{3}\langle \tfrac32,-\tfrac12|O_s|\tfrac32,-\tfrac12\rangle - \frac{\sqrt2}{2}\langle \tfrac12,-\tfrac12|O_s|\tfrac12,-\tfrac12\rangle = \frac{\sqrt2}{3}A_{3/2} - \frac{\sqrt2}{3}A_{1/2} = \frac{1}{\sqrt2}\left(S_{\pi^+ p} - S_{\pi^- p}\right)$

EKSPERIMENT: AKCELERATORI I DETEKTORI

- dvije najvažnije značajke svakog akceleratora su energija u SCM, koja nam određuje tipove čestica koje možemo proizvesti ($\sqrt{s} = E_{cm}$), i luminozitet $\mathcal{L}$ koji nam određuje učestalost (ili rate) događaja (broj događaja po jedinici vremena)
- trenutni luminozitet je dan formulom:

$$N = \sigma \int \mathcal{L}(t)\, dt \quad \Rightarrow \quad \mathcal{L}(t) = \frac{1}{\sigma} \frac{dN}{dt}$$

Broj interakcija

udarni presjek. Određen FEČ; proporcionalan $|\langle f|i\rangle|^2$

Luminozitet
dan izvedbom eksperimenta

- ovo nam je poveznica između eksperimenta i teorije

- u akceleratorima čestice su grupirane u nakupine koje se sudaraju u jednoj ili više točaka duž akceleratora, gdje se nalaze detektori
- u LHC-u nakupine su udaljene 25 ns što odgovara frekvenciji sudara od $f = 40\,MHz$
- trenutni luminozitet se može izraziti kao

$$\mathcal{L}(t) = f \frac{n_1 n_2}{4\pi \sigma_x \sigma_y}$$

- napomena: rms
$x_{rms} = \sqrt{\frac{1}{T_1 - T_0} \int_{T_0}^{T_1} [f(t)]^2 dt}$
- površina elipse
$\pi a b$
ukupni luminozitet
$L = \frac{N}{\sigma} \Rightarrow L = \frac{N_1 N_2}{A}$

n_1, n_2 – broj čestica po nakupini

σ_x, σ_y – root mean square (rms) horizontalna i vertikalna veličina snopa

- pretpostavka je da snopovi imaju Gaussov profil i da se sudaraju čeono

Zadatak

Na sudaraču LEP, opsega 27km, elektroni i pozitroni su smješteni u 4 jednako razmaknute nakupine (bunches) od kojih svaka daje snopu struju od 1.0 mA

Snopovi se sudaraju čeono na točki interakcije gdje snop ima profil rms $\sigma_x = 250\,\mu m$ i $\sigma_y = 4\,\mu m$

$R = 27\,km$
broj nakupina $= 4$
$I/\text{nakupini} = 1.0\,mA$
$\sigma_x = 250\,\mu m$
$\sigma_y = 4\,\mu m$

a) Odredite trenutni luminozitet:

$$\mathcal{L} = f \frac{n_1 n_2}{4\pi \sigma_x \sigma_y}$$

- za n_e elektrona/pozitrona u nakupini ("bunchu") koja kruži frekvencijom $f_b = \frac{c}{27 \cdot 10^3} = 11.1\,kHz$, struja nakupine je $I = f_b \cdot n_e e$
- zadano je $I = 1\,mA \Rightarrow n_e = 5.6 \cdot 10^{11} \Rightarrow (n_1 = n_2 = 5.6 \cdot 10^{11})$
- za 4 nakupine po snopu frekvencija sudaranja nakupina je $f = 4 \cdot \frac{c}{27 \cdot 10^3} = 4 f_b = 44.4\,kHz$

Trenutna luminoznost je $\mathcal{L} = 44.4\cdot10^{3}\,\frac{(5,6\cdot10^{11})^{2}}{4\pi\cdot10^{-5}} = 1,1\cdot10^{32}\,cm^{-2}s^{-1}$

b) Učestalost događaja („event rate") procesa $e^+e^- \to Z$ koji ima udarni presjek $\sigma \simeq 40\,nb$

$\sigma = 40\,nb = 4\cdot10^{-8}\cdot10^{-24}\,cm^2$

- učestalost događaja

$\sigma\mathcal{L} = 4\cdot10^{-32}\cdot10^{32}\,s^{-1} = 4\,s^{-1}$

- čestice pri prolazu kroz materijal (detektor) gube energiju

Zadatak

Visoko energetski mioni pri prolazu kroz materijal gube energiju prema formuli:

$-\frac{1}{\rho}\frac{dE}{dx} \approx a + bE$

- a dolazi od ionizacije, a b opisuje zakočno zračenje i tvorbu e^+e^- parova. Za uobičajeni standardni kamen (recimo $A=22$, $Z=11$ i $\rho=2,65\,g\,cm^{-3}$) parametri a i b slabo ovise o energiji miona i imaju vrijednosti $a \approx 2,5\,MeV\,g^{-1}cm^{2}$ $b \approx 3,5\cdot10^{-6}\,g^{-1}cm^{2}$

a) Na kojoj energiji ionizacije i zakočno zračenje ti tvorba parova doprinose jednako procesu gubitka energije?

$a = bE_c \Rightarrow E_c = \frac{a}{b} = 714285\,MeV = 714\,GeV$

b) Kolika udaljenost prođe mion energije $E_0 = 100\,GeV$?

Opće homogeno rješenje:

$\frac{dE}{dx} = -\rho bE \Rightarrow \int\frac{dE}{E} = -\int\rho b\,dx \Rightarrow E = Ae^{-\rho bx}$

Partikularno rješenje:

$E = -\frac{a}{b} \Rightarrow E(x) = Ae^{-\rho bx} - \frac{a}{b}$

$E(x=0) = 100\,GeV = A - \frac{a}{b} \Rightarrow A = (100+714)\,GeV = 814\,GeV$

$E(x) = 0 = 814e^{-\rho bx} - 714 \Rightarrow 814e^{-\rho bx} = 714$

$e^{-\rho bx} = \frac{714}{814}$ / ln

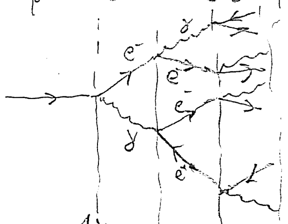
$\rho bx = -\ln\frac{714}{814}$

$x = -\frac{1}{\rho b}\ln\frac{714}{814} = 14132\,cm = 141\,m$

Elektromagnetski pljuskovi

Bremsstrahlung

Kad visoko energetski elektron interagira s nukleonom on zrači visoko energetski foton koji zatim producira e^+e^- par i tako dalje



$1X_0$ $2X_0$ $3X_0$ $4X_0$

X_0 = radijacijska duljina

- nakon svake radijacijske duljine broj čestica se udvostruči
- nakon n radijacijskih duljina prosječna energija čestica je $\langle E \rangle \approx \frac{E_0}{2^n}$
- pljusak se razvija do kritične energije E_c kada dalji proces gubitka energije postaje ionizacija
- dalje maksimalan broj čestica se javlja nakon n_{max} radijacijskih duljina kada vrijedi $\langle E \rangle \approx E_c$

$$n_{max} = \frac{\ln(E_0/E_c)}{\ln 2}$$

<u>Zadatak</u> Volfram ima radijacijsku duljinu $X_0 = 0{,}35$ cm i kritičnu energiju $E_c = 7{,}97$ MeV. Koliko debljina volframa je potrebna da u potpunosti apsorbira elektromagnetski pljusak uzrokovan elektronom energije 500 MeV

$d = ?$

$X_0 = 0{,}35$ cm

$E_c = 7{,}97$ MeV

$E_0 = 500$ GeV $= 500 \cdot 10^3$ MeV

$$n_{max} = \frac{\ln\left(\frac{E_0}{E_c}\right)}{\ln 2} \approx 16$$

$d \approx n_{max} \cdot X_0 = \underline{5{,}6 \text{ cm}}$

DIRACOVA JEDNADŽBA

Uvod:

- perskripcija za dobivanje Schrödingerove jednadžbe za slobodnu česticu
- u izrazu za nerelativističku energiju:

$$E=\frac{p^2}{2m} \quad ; \text{ napravimo supstitucije: } \quad E\to i\hbar\frac{\partial}{\partial t}, \quad \vec{p}\to i\hbar\vec{\nabla}$$

- dobivenu jednadžbu interpretiramo kao operator koji djeluje na kompleksnu valnu funkciju $\psi(\vec{x},t)$ [uz $\hbar=1$]

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2m}\nabla^2\psi=0$$

- $\rho=|\psi|^2$ - predstavlja gustoću vjerojatnosti
- $|\psi|^2 d^3x$ - daje vjerojatnost nalaženja čestice u elementu volumena d^3x
- jednadžba kontinuiteta

$$i\frac{\partial\psi}{\partial t}+\frac{1}{2m}\nabla^2\psi=0 \;/(-i\psi^*)$$
$$-i\frac{\partial\psi^*}{\partial t}+\frac{1}{2m}\nabla^2\psi^*=0 \;/(-i\psi)$$

$$/- \Rightarrow \psi^*\frac{\partial\psi}{\partial t}+\frac{\partial\psi^*}{\partial t}\psi-\frac{i}{2m}\left(\psi^*\nabla^2\psi-\psi\nabla^2\psi^*\right)=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\vec{j}=0 \quad ; \quad \rho=\psi^*\psi, \quad \vec{j}=-\frac{i}{2m}\left(\psi^*\nabla\psi-\psi\nabla\psi^*\right)$$

- to je ok, to sve funkcionira!
- za ravni val

$$\psi=Ne^{i\vec{p}\cdot\vec{x}-iEt} \Rightarrow \rho=|N|^2 \; ; \; \vec{j}=\frac{\vec{p}}{m}|N|^2=\rho\vec{v}$$

Klein-Gordonova jednadžba (prvi pokušaj relativističke kvantne fizike):

- u izrazu za relativističku energiju:

$$E^2=\vec{p}^2+m^2 \quad ; \text{ napravimo iste supstitucije: } \quad E\to i\hbar\frac{\partial}{\partial t}, \quad \vec{p}\to i\hbar\vec{\nabla}$$

- dobivamo:

$$-\frac{\partial^2}{\partial t^2}\phi=(-\nabla^2+m^2)\phi$$ ili uvođenjem D'Alembertiana $\Box\equiv\partial_\mu\partial^\mu=\frac{\partial^2}{\partial t^2}-\nabla^2$

$$(\Box+m^2)\phi=0$$ - eksplicitno kovarijantno

- jednadžba kontinuiteta (isti postupak kao i kod Schrödingerove)

$$(-i\phi^*)/\ \frac{\partial^2}{\partial t^2}\phi-\nabla^2\phi+m^2\phi=0$$
$$\frac{\partial^2\phi^*}{\partial t^2}-\nabla^2\phi^*+m^2\phi^*=0 \;/(i\phi)$$

$$/- \Rightarrow \frac{\partial}{\partial t}\Big[\underbrace{i\left(\phi^*\frac{\partial\phi}{\partial t}-\phi\frac{\partial\phi^*}{\partial t}\right)}_{\rho}\Big]+\vec{\nabla}\cdot\Big[\underbrace{-i\left(\phi^*\vec{\nabla}\phi-\phi\vec{\nabla}\phi^*\right)}_{\vec{j}}\Big]=0$$

- za slobodnu česticu:

$$\phi = N e^{i\vec{p}\vec{x} - iEt}$$

$$\Rightarrow \quad \rho = 2E|N|^2 \; ; \; \vec{j} = 2\vec{p}|N|^2$$

Problemi K-G jednadžbe:

1. K-G jednadžba vodi na $E = \pm(\vec{p}^2 + m^2)^{1/2}$. Negativne energije koje ulaze u potpuni skup stanja. Prijelazak na sve niže energije. $\hat{E}$ nije ograničeno odozdo.
2. Negativnim energijama pridružene su negativne vjerojatnosti

$\underline{E < 0 \text{ vrijedi tada } \rho < 0}$

- povijesno je to vodilo na odbacivanje K-G jednadžbe (u stvarnosti će se uvidjeti da K-G jed. opisuje čestice spina = 0)

Diracova jednadžba:

- problem negativnih vjerojatnosti je uzrokovan time što je K-G jed. drugog reda u vremenu
- pokušava se naći relativistički kovarijantna valna jednadžba Schrödingerovog tipa (prvog reda u vremenu)
- ironično to ne rješava problem negativnih energija.

$$i\frac{\partial\psi}{\partial t} = \left[\frac{1}{i}\left(\hat{\alpha}_1\frac{\partial}{\partial x_1} + \hat{\alpha}_2\frac{\partial}{\partial x_2} + \hat{\alpha}_3\frac{\partial}{\partial x_3}\right) + \hat{\beta}m\right]\psi = \hat{H}_D\psi$$

- koeficijenti α_i, β ne mogu biti obični brojevi (jednadžba ne bi bila invarijantna na rotacije)
- $\hat{\alpha}_i, \hat{\beta}$ su matrice, a $\psi(x,t)$ vektor (spinor) $\psi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \\ \vdots \end{pmatrix}$ tako da bi smo imali sustav vezanih dif. jednadžbi
- na jednadžbu postavljamo 3 prirodna uvjeta:
  a) da zadovoljava relativističku vezu:
  $E^2 = p^2 + m^2$
  b) da zadovoljava jednadžbu kontinuiteta za gustoću $\varsigma = \psi^+\psi$
  c) Lorentz invarijantnost

a) - svaka komponenta ψ_α mora zasebno zadovoljavati K-G jednadžbu
- u Klein-Gordonovoj jednadžbi javlja nam se $\frac{\partial^2}{\partial t^2}$:

$$\left[i\frac{\partial}{\partial t} + \left(\frac{1}{i}\hat{\alpha}_k\frac{\partial}{\partial x_k} + \hat{\beta}m\right)\right]\left[i\frac{\partial}{\partial t} - \left(\frac{1}{i}\hat{\alpha}_k\frac{\partial}{\partial x_k} + \hat{\beta}m\right)\right]\psi = 0$$

$$-\frac{\partial^2\psi}{\partial t^2} - \left(\frac{1}{i}\hat{\alpha}_k\frac{\partial}{\partial x_k} + \hat{\beta}m\right)^2\psi = 0$$

$$-\frac{\partial^2\psi}{\partial t^2} + \sum_{k,k'=1}^{3}\frac{\left(\hat{\alpha}_k\hat{\alpha}_{k'} + \hat{\alpha}_{k'}\hat{\alpha}_k\right)}{2}\frac{\partial^2\psi}{\partial x_k\partial x_{k'}} - \frac{m}{i}\sum_{k=1}^{3}\left(\hat{\alpha}_k\hat{\beta} + \hat{\beta}\hat{\alpha}_k\right)\frac{\partial\psi}{\partial x_k} - \hat{\beta}^2m^2\psi = 0$$

- uspoređujući s K-G:

$$\left(-\frac{\partial^2}{\partial t^2} + \nabla^2 - m^2\right)\phi = 0$$

- vidimo da mora vrijediti: (antikomutacijske jednadžbe):

$$\hat{\alpha}_i\hat{\alpha}_j + \hat{\alpha}_j\hat{\alpha}_i = 2\delta_{ij}$$
$$\hat{\alpha}_i\hat{\beta} + \hat{\beta}\hat{\alpha}_i = 0$$
$$\hat{\alpha}_i^2 = \hat{\beta}^2 = \mathbb{1}$$

- da bi hamiltonijan bio hermitski operator (Hamiltonijan mora biti hermitski operator):

$$\Rightarrow \hat{\alpha}_i = \hat{\alpha}_i^+ \; , \; \hat{\beta} = \hat{\beta}^+$$

- Budući da imamo $\hat{\alpha}_i^2 = \mathbb{1}$ i $\hat{\beta}_i^2 = \mathbb{1}$
  $\Rightarrow$ svojstvene vrijednosti su ± 1

Napomena: eigenvrijednosti ne ovise o reprezentaciji
$A\Psi_\alpha = \alpha \Psi_\alpha \Rightarrow UAU^{-1}U\Psi_\alpha = \alpha U\Psi_\alpha \Rightarrow A' U\Psi_\alpha = \alpha U\Psi_\alpha$

- s druge strane trag matrica $\hat{\alpha}_i$ i $\hat{\beta}$ mora biti nula.
  $\hat{\alpha}_i = -\hat{\beta}\hat{\alpha}_i\hat{\beta}$; $\hat{\beta}^2 = \mathbb{1}$; $\hat{\alpha}_i^2 = \mathbb{1}$
  $tr\hat{\alpha}_i = tr\hat{\beta}^2\hat{\alpha}_i = tr\hat{\beta}\hat{\alpha}_i\hat{\beta} = -tr\hat{\alpha}_i \Rightarrow tr\hat{\alpha}_i = 0$ i slično za $\hat{\beta}$

za trag
$trA = trU^{-1}UA = trUAU^{-1} = trA'$

- Trag je umnožak zbroj eigen vrijednosti
  $\Rightarrow$ eigenvrijednosti ± 1, a njihov zbroj $0 \Rightarrow$ mora dim biti paran broj (2-dim, 4-dim, 6-dim matrice)
- za $D=2$ postoje samo tri antikomutirajuće matrice (Paulijeve matrice), nama trebaju 4.
- idemo na $D=4$
- postoji više izbora.
- svi valjaju i daju iste fiziku u konačnici
- Biramo Diracovu reprezentaciju

$$\hat{\alpha}_i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ \hat{\sigma}_i & 0 \end{pmatrix} ; \quad \hat{\beta} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \qquad \hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} ; \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Diracovu jednadžbu je moguće staviti u zgodniji oblik:

$\hat{\beta}\ /\ \left(i\frac{\partial}{\partial t} + i\hat{\alpha}_i \frac{\partial}{\partial x_i} - \hat{\beta}m\right)\Psi = 0$

i definiramo $\gamma^0 = \hat{\beta}$; $\gamma^i = \hat{\beta}\hat{\alpha}_i$

$\Rightarrow (i\gamma^\mu \partial_\mu - m)\Psi = 0$

- antikomutacijske relacije postaju
  $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}$ – ovo je antikomutacija $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$; $g^{\mu\nu}$ nije matrica nego matrični element

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} ; \quad \gamma^i = \begin{pmatrix} 0 & \hat{\sigma}_i \\ -\hat{\sigma}_i & 0 \end{pmatrix}$$

b) Pokazati jednadžbu kontinuiteta

$$i\Psi^\dagger \frac{\partial}{\partial t}\Psi = \frac{1}{i}\Psi^\dagger \hat{\alpha}_k \frac{\partial \Psi}{\partial x_k} + m\Psi^\dagger \beta \Psi$$
$$-i\frac{\partial \Psi^\dagger}{\partial t}\Psi = -\frac{1}{i}\frac{\partial \Psi^\dagger}{\partial x_k}\hat{\alpha}_k \Psi + m\Psi^\dagger \beta \Psi$$

$$\Rightarrow \quad i\frac{\partial}{\partial t}(\Psi^\dagger\Psi) = \frac{1}{i}\frac{\partial}{\partial x_k}(\Psi^\dagger \hat{\alpha}_k \Psi)$$

$\rho = \Psi^\dagger\Psi$, $\vec{j} = \Psi^\dagger \vec{\alpha}\Psi$; $\vec{\alpha} = (\hat{\alpha}^1, \hat{\alpha}^2, \hat{\alpha}^3)$

c) Lorentz invarijantnost ostavljamo za idući sat.

- sada se povećajmo izrazima s gama γ matricama (važni pri računu amplituda raspršenja)

1) iz antikomutacijskih relacija tražimo
$(\gamma^0)^2 = \mathbb{1}$ $\quad (\gamma^i)^2 = -\mathbb{1}$

2) u FEČ važna je i tzv. γ^5 matrica $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \underbrace{(\gamma^0)^2}_{\mathbb{1}}\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \underbrace{(\gamma^1)^2}_{-\mathbb{1}}\gamma^2\gamma^3\gamma^2\gamma^3 = +\underbrace{(\gamma^2)^2}_{-\mathbb{1}}\underbrace{(\gamma^3)^2}_{-\mathbb{1}} = \mathbb{1}$$

3). $\gamma^\mu\gamma^5 = i\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3 = \left[\text{za } \mu=2\right] = i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -\gamma^5\gamma^2$

$\Rightarrow \gamma^\mu\gamma^5 = -\gamma^5\gamma^\mu \Rightarrow \{\gamma^\mu,\gamma^5\}=0$

4). Pogledajmo sada svojstva γ matrica na hermitsko konjugiranje

$$\gamma^{0\dagger} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \gamma^0 = \gamma^0\gamma^0\gamma^0$$

$$\gamma^{i\dagger} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & -\sigma_i^\dagger \\ \sigma_i^\dagger & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} = -\gamma^i = \gamma^0\gamma^i\gamma^0$$

$$\gamma^{5\dagger} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \gamma^5 \qquad \boxed{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \ ; \ \gamma^{5\dagger} = \gamma^5}$$

5) $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}\mathbb{1} \ / \cdot g_{\mu\nu}$

$2\gamma^\mu\gamma_\mu = 2g^\mu{}_\mu\mathbb{1} \Rightarrow \gamma_\mu\gamma^\mu = 4\cdot\mathbb{1}$

[tu smo uveli $\gamma_\mu = \gamma^\nu g_{\nu\mu}$

6) $\gamma_\mu\gamma^\nu\gamma^\mu = \gamma_\mu(2g^{\mu\nu} - \gamma^\mu\gamma^\nu) = 2\gamma^\nu - \underbrace{\gamma_\mu\gamma^\mu}_{4\mathbb{1}}\gamma^\nu = -2\gamma^\nu$

7) $\gamma_\mu\gamma^\sigma\gamma^\tau\gamma^\mu = \gamma_\mu\gamma^\sigma(2g^{\tau\mu} - \gamma^\mu\gamma^\tau) = 2\underbrace{g^{\tau\mu}\gamma_\mu}_{\gamma^\tau}\gamma^\sigma - \underbrace{\gamma_\mu\gamma^\sigma\gamma^\mu}_{-2\gamma^\sigma}\gamma^\tau = 2\underbrace{(\gamma^\tau\gamma^\sigma + \gamma^\sigma\gamma^\tau)}_{2g^{\sigma\tau}} = 4g^{\sigma\tau}$

8) Svojstva tragova

Trag od neparnog broja γ matrica uvijek iščezava

n-neparan broj

$Tr[\gamma^{\mu_1}\gamma^{\mu_2}\gamma^{\mu_3}\dots\gamma^{\mu_n}] = Tr[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\underbrace{\gamma^5\gamma^5}_{\mathbb{1}}] = -Tr[\gamma^5\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^5] \overset{\text{cikličko svojstvo}}{=} -Tr[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}\gamma^5\gamma^5] = -Tr[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}]$

dakle $Tr[\gamma^{\mu_1}\gamma^{\mu_2}\dots\gamma^{\mu_n}]=0$

9) $Tr[\gamma^\mu\gamma^\nu] = \frac{1}{2}\left(Tr[\gamma^\mu\gamma^\nu] + Tr[\gamma^\nu\gamma^\mu]\right) = \frac{1}{2}Tr[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] \overset{\text{antikom. relacija}}{=} \frac{1}{2}Tr[2g^{\mu\nu}\mathbb{1}] = g^{\mu\nu}Tr[\mathbb{1}] = 4g^{\mu\nu}$

c) Lorentz invarijantnost Diracove jednadžbe (tražimo kako se transformira ψ)

- sustav S

$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$

- sustav S'

$(i\gamma^\mu\frac{\partial}{\partial x'^\mu} - m)\psi'(x') = 0$

$\Lambda/ \ (i\gamma^\mu\Lambda^\nu{}_\mu\partial'_\nu - m)S^{-1}(\Lambda)\psi'(x') = 0$

$[iS(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu\partial'_\nu - m]\psi'(x') = 0$

$\Rightarrow S^{-1}/ \ S(\Lambda)\gamma^\mu S^{-1}(\Lambda)\Lambda^\nu{}_\mu = \gamma^\nu \ / \ S$

$\gamma^\mu\Lambda^\nu{}_\mu = \Lambda_\mu{}^\nu\gamma^\mu = S^{-1}(\Lambda)\gamma^\nu S(\Lambda)$

- definirajuća relacija

- za paritet (koji spada u Lorentzove transformacije). imamo: $(\Lambda)^\nu{}_\mu = \begin{pmatrix} 1&0&0&0 \\ 0&-1&0&0 \\ 0&0&-1&0 \\ 0&0&0&-1 \end{pmatrix}$

$S_p^{-1}\gamma^\nu S_p = (\Lambda)^\nu{}_\mu\gamma^\mu \Rightarrow \begin{matrix} S_p^{-1}\gamma^i S_p = (-1)\gamma^i \\ S_p^{-1}\gamma^0 S_p = \gamma^0 \end{matrix} \Rightarrow \begin{matrix} [\gamma^0, S_p] = 0 \\ \{\gamma^i, S_p\} = 0 \end{matrix} \quad \boxed{S_p = \gamma^0}$

- općenito za boostove izvod zahtjeva malo matematičkog eqzibicionizma
- W. Greiner – Relativistic QM, ch. 3 & 6
- ovdje dajemo samo konačni rezultat

$$\hat{S}(-\vec{p}) = \sqrt{\frac{E+m}{2m}} \begin{pmatrix} 1 & 0 & \frac{p_z}{E+m} & \frac{p_-}{E+m} \\ 0 & 1 & \frac{p_+}{E+m} & \frac{-p_z}{E+m} \\ \frac{p_z}{E+m} & \frac{p_-}{E+m} & 1 & 0 \\ \frac{p_+}{E+m} & \frac{-p_z}{E+m} & 0 & 1 \end{pmatrix} \quad ; \quad p_\pm = p_x \pm i p_y$$

- također vrijedi: (ovo će studenti pokazati za zadaću)

$$\hat{S}^{-1} = \gamma^0 \hat{S}^\dagger \gamma^0$$

- uvodimo bilinearne kovarijante:
- def. $\bar{\Psi} = \Psi^\dagger \gamma^0$; $\Psi^\dagger = \Psi^{*T}$
- transformacija $\bar{\Psi} \Rightarrow \bar{\Psi}' = \Psi'^\dagger \gamma^0 = (\hat{S}\Psi)^\dagger \gamma^0 = \Psi^\dagger \hat{S}^\dagger \gamma^0 = \Psi^\dagger \gamma^0 \underbrace{\gamma^0 \hat{S}^\dagger \gamma^0}_{\hat{S}^{-1}} = \bar{\Psi} \hat{S}^{-1}$

Bilinearne kovarijante:

1) $\bar{\Psi}\Psi \Rightarrow \bar{\Psi}'\Psi' = \bar{\Psi}\hat{S}^{-1}\hat{S}\Psi = \bar{\Psi}\Psi$ – skalar na L. transf.

2) $\bar{\Psi}\gamma^5\Psi \longrightarrow \bar{\Psi}'\gamma^5\Psi' = \bar{\Psi}\underbrace{S^{-1}\gamma^5 S}_{\text{za kont. transformaciju (DZ) } [\gamma^5, S] = 0}\Psi = \bar{\Psi}\gamma^5\Psi$

- za paritet $S_p = \gamma_0$; $S_p^{-1} = \gamma_0 \Rightarrow \gamma_0\gamma^5\gamma_0 = -\gamma^5$, pa slijedi $P: \bar{\Psi}\gamma^5\Psi = \bar{\Psi}\gamma_0\gamma^5\gamma_0\Psi = -\bar{\Psi}\gamma^5\Psi$ – pseudoskalar

3) $\bar{\Psi}\gamma^\mu\Psi \to \bar{\Psi}'\gamma^\mu\Psi' = \bar{\Psi}\underbrace{S^{-1}\gamma^\mu S}_{\Lambda^\mu{}_\nu\gamma^\nu}\Psi = \bar{\Psi}\Lambda^\mu{}_\nu\gamma^\nu\Psi = \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\Psi$ – vektor

- za paritet

$$\bar{\Psi}\gamma^\mu\bar{\Psi} \to \bar{\Psi}S_p^{-1}\gamma^\mu S_p\Psi = \bar{\Psi}\gamma^0\gamma^\mu\gamma^0\Psi = \begin{cases} \mu = 0 & \bar{\Psi}\gamma^0\Psi \\ \mu = i & -\bar{\Psi}\gamma^i\Psi \end{cases}$$

4) $\bar{\Psi}\gamma^\mu\gamma^5\Psi \to \bar{\Psi}'\gamma^\mu\gamma^5\Psi' = \bar{\Psi}S^{-1}\gamma^\mu\gamma^5 S\Psi = \Lambda^\mu{}_\nu\bar{\Psi}\gamma^\nu\gamma^5\Psi$

a) $S^{-1}\gamma^\mu\gamma^5 S = S^{-1}\gamma^\mu S\gamma^5 = \Lambda^\mu{}_\nu\gamma^\nu\gamma^5$

b) $S_p^{-1}\gamma^\mu\gamma^5 S_p = \gamma^0\gamma^\mu\gamma^5\gamma^0 = -\gamma^0\gamma^\mu\gamma^0\gamma^5 = \begin{cases} \mu = 0, & -\gamma^0\gamma^5 \\ \mu = i, & \gamma^i\gamma^5 \end{cases}$ – pseudovektor
– vremenska komponenta mijenja predznak

5) uvodimo $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$

$$\bar{\Psi}\sigma^{\mu\nu}\Psi \to \frac{i}{2}\bar{\Psi}S^{-1}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)S\Psi = \frac{i}{2}\bar{\Psi}(S^{-1}\gamma^\mu S S^{-1}\gamma^\nu S - S^{-1}\gamma^\nu S S^{-1}\gamma^\mu S)\Psi = \frac{i}{2}\bar{\Psi}(\Lambda^\mu{}_\sigma\gamma^\sigma\Lambda^\nu{}_\varrho\gamma^\varrho - \Lambda^\mu{}_\varrho\gamma^\varrho\Lambda^\mu{}_\sigma\gamma^\sigma)\Psi =$$

$$= \Lambda^\mu{}_\sigma\Lambda^\nu{}_\varrho\bar{\Psi}\sigma^{\sigma\varrho}\Psi \qquad \text{– Tenzor}$$

- sve skupa 16 bilinearnih kovarijanti koje se lijepo transformiraju (umjesto $\bar{\Psi}_i\Psi_j \to 16$ koje se svakako...)

- Rješenja Diracove jednadžbe [napomena: slash notacija $\gamma^\mu a_\mu = \not{a}$, a_μ - vektor]

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\left(i\frac{\partial}{\partial t} + i\hat{\alpha}_i \frac{\partial}{\partial x_i} - \hat{\beta} m\right)\psi(x) = 0$$

$$\psi(x) = \psi(\vec{x})e^{-ip^0 t}$$

$$\left(p^0 + i\hat{\alpha}_i \frac{\partial}{\partial x_i} - \hat{\beta} m\right)\psi(\vec{x}) = 0$$

$$\psi(\vec{x}) = \omega(p)e^{+i\vec{p}\vec{x}}$$

$$p^0\omega(p) = \begin{pmatrix} m\mathbb{1} & \vec{\sigma}\vec{p} \\ \vec{\sigma}\vec{p} & -m\mathbb{1} \end{pmatrix}\omega(p)$$

- u sustavu mirovanja $\vec{p} = 0$ imamo četiri rješenja

1) $p^0 = m$ $\quad \omega_1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$; $\omega_2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $\quad$ tj. $\psi(x) = \omega_r(0)e^{-i\varepsilon_r m t}$; $\varepsilon_r = \begin{cases} 1, & r = 1,2 \\ -1, & r = 3,4 \end{cases}$

2) $p^0 = -m$ $\quad \omega_3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$; $\omega_4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- u sustavu koji se giba $\vec{p} \neq 0$ možemo pisati:

$\omega(p) = \begin{pmatrix} \chi \\ \varrho \end{pmatrix}$; χ, ϱ - dvokomponentni

dobivamo

$$p^0\chi = m\chi + \vec{\sigma}\vec{p}\varrho \Rightarrow (p^0 - m)\chi = \vec{\sigma}\vec{p}\varrho$$
$$p^0\varrho = \vec{\sigma}\vec{p}\chi - m\varrho \Rightarrow (p^0 + m)\varrho = \vec{\sigma}\vec{p}\chi$$

- netrivijalna rješenja postoje ako determinanta iščezava

$$\begin{vmatrix} (p^0 - m) & -\vec{\sigma}\vec{p} \\ -\vec{\sigma}\vec{p} & (p^0 + m) \end{vmatrix} = 0 = p^{02} - m^2 - (\vec{\sigma}\vec{p})^2 = p^{02} - m^2 - \vec{p}^2 \Rightarrow p^{02} = \vec{p}^2 + m^2$$
$$p^0 = \pm\sqrt{\vec{p}^2 + m^2} = \pm E$$

$$(\vec{\sigma}\vec{p})^2 = \vec{p}^2$$

iz relacije

$$(\vec{\sigma}\vec{A})(\vec{\sigma}\vec{B}) = \vec{A}\vec{B}\mathbb{1} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$$

- za $E > 0$ uzmimo $\chi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ - dva linearno nezavisna vektora (jedan mogući izbor)

$\Rightarrow \varrho = \frac{\vec{\sigma}\vec{p}}{E + m}\chi \Rightarrow$

$$\omega_1(p) = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} ; \quad \omega_2(p) = \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

- za $E < 0$ uzmimo $\varrho = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \chi = \frac{\vec{\sigma}\vec{p}}{-E-m}\varrho = -\frac{\vec{\sigma}\vec{p}}{E+m}\varrho \Rightarrow \omega_3(p) = \begin{pmatrix} -\frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \quad \omega_4(p) = \begin{pmatrix} -\frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

- drugi način da ovo dobijemo je da jednostavno boostiramo rješenja za slobodnu česticu $\omega_r(0)$
- stupci matrice $\hat{S}(-\beta)$ su upravo $\omega_r(p)$ do na jednu malu razliku.
- u ω_3 i ω_4 je drugačiji predznak od $\vec{p}$
- ovo ima veze s time da elektron energije $-E$ i impulsa $-\vec{p}$ i spina $-1/2$ interpretiramo kao pozitron energije $+E$, impulsa $+\vec{p}$ i spina $+1/2$
- stoga se uvode oznake

$$\omega_{1,2}(p)e^{-ipx} \equiv u^{1,2}(p)e^{-ipx} \quad ; \quad \omega_{3,4}(-p)e^{-i(-p)\cdot x} \equiv v^{2,1}(p)e^{ip\cdot x} \quad \text{gdje je } p^0 \equiv +E > 0$$

- Relacije s u i v:
Diracove jednadžbe

$(\not p - m)u = 0$ hermitski konjugirano

$(\not p + m)v = 0$

$$(\gamma^\mu p_\mu u - mu)^\dagger = 0 \Rightarrow u^\dagger \gamma^{\mu\dagger} p_\mu - m u^\dagger = 0 \,\big|\gamma^0 \Rightarrow (\underbrace{u^\dagger\gamma^0}_{\bar u}\gamma^\mu p_\mu - m u^\dagger\gamma^0) = 0$$

$$\bar u(\not p - m) = 0$$
$$\bar v(\not p + m) = 0$$

- Normalizacija:
- volumni element $\Delta V = \Delta x \Delta y \Delta z$ nije Lorentz invarijantan
$\Rightarrow$ za boost u x smjeru $\Delta V' = \Delta x' \Delta y' \Delta z' = \Delta x \underbrace{\sqrt{1-\beta^2}}_{} \Delta y \Delta z$
$\Rightarrow$ stoga normalizirati na 1 česticu u jediničnom (kontrakcija) volumenu nije Lorentz invarijantno
- s druge strane $2E\Delta V$ je Lorentz invarijantno jer $2E = \frac{2m}{\sqrt{1-\beta^2}}$
- normaliziramo:

$$\int \psi^\dagger\psi \, d^3r = 2EV \Rightarrow u^\dagger u = 2E \text{ za } u^1 \Rightarrow N^2\left((1\ 0)\ \underbrace{\frac{\vec\sigma\vec p}{E+m}(1\ 0)}\right)\begin{pmatrix}\binom{1}{0}\\ \frac{\vec\sigma\vec p}{E+m}\binom{1}{0}\end{pmatrix} = N^2\left(1+\frac{(\vec\sigma\vec p)^2}{(E+m)^2}\right) = N^2\left(1+\frac{\vec p^2}{(E+m)^2}\right)$$

$$= N^2\frac{E^2+2Em+\overbrace{m^2+\vec p^2}^{E^2}}{(E+m)^2} = N^2\frac{2E(E+m)}{(E+m)^2} = N^2\frac{2E}{E+m} \Rightarrow N^2 = E+m;\ N = \sqrt{E+m}$$

tako da $u^1(p) = \sqrt{E+m}\begin{pmatrix}\binom{1}{0}\\ \frac{\vec\sigma\vec p}{E+m}\binom{1}{0}\end{pmatrix}$ ...

- možemo odmah pisati
1) $u^{(r)\dagger}u^{(s)} = 2E\,\delta_{rs}$
$v^{(r)\dagger}v^{(s)} = 2E\,\delta_{rs}$
- također vrijedi
2) $\bar u^{(s)}u^{(s)} = 2m$
$\bar v^{(s)}v^{(s)} = -2m$

Primjer: u^1

$$\bar u^1 u^1 = u^{1\dagger}\gamma^0 u^1 = (E+m)\left((1\ 0)\ \frac{\vec\sigma\vec p}{E+m}(1\ 0)\right)\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\binom{1}{0}\\ \frac{\vec\sigma\vec p}{E+m}\binom{1}{0}\end{pmatrix} =$$

$$= (E+m)\left(1-\frac{(\vec\sigma\vec p)^2}{(E+m)^2}\right) = (E+m)\frac{E^2+2Em+m^2-\vec p^2}{(E+m)^2} = \frac{2m\cancel{(E+m)}}{E+m} = \underline{2m}$$

3) $$\sum_{s=1,2} u(p,s)\bar u(p,s) = ? \quad \text{(Relacija potpunosti)}$$

$$u(p,1)\bar u(p,1) = (E+m)\begin{pmatrix}\binom{1}{0}\\ \frac{\vec\sigma\vec p}{E+m}\binom{1}{0}\end{pmatrix}\left((1\ 0)\ \frac{\vec\sigma\vec p}{E+m}(1\ 0)\right)\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix} = (E+m)\begin{pmatrix}\binom{1}{0}\\ \frac{\vec\sigma\vec p}{E+m}\binom{1}{0}\end{pmatrix}\left((1\ 0)\ \ -\frac{\vec\sigma\vec p}{E+m}(1\ 0)\right) =$$

$$= (E+m)\begin{pmatrix}\begin{pmatrix}1&0\\0&0\end{pmatrix} & -\frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}1&0\\0&0\end{pmatrix}\\ \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}1&0\\0&0\end{pmatrix} & -\frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{\vec\sigma\vec p}{E+m}\end{pmatrix}$$

$$u(p,2)\bar u(p,2) = (E+m)\begin{pmatrix}\begin{pmatrix}0&0\\0&1\end{pmatrix} & -\begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{\vec\sigma\vec p}{E+m}\\ \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}0&0\\0&1\end{pmatrix} & -\frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{\vec\sigma\vec p}{E+m}\end{pmatrix}$$

$$\Rightarrow \sum_{s=1,2} u(p,s)\bar u(p,s) = (E+m)\begin{pmatrix}\mathbb{1} & -\frac{\vec\sigma\vec p}{E+m}\\ \frac{\vec\sigma\vec p}{E+m} & -\frac{(\vec\sigma\vec p)^2}{(E+m)^2}\end{pmatrix} = \begin{pmatrix}(E+m)\mathbb{1} & -\vec\sigma\vec p\\ \vec\sigma\vec p & -\frac{\vec p^2}{E+m}\end{pmatrix} - m\mathbb{1}_{4\times4} + m\mathbb{1}_{4\times4} =$$

$$= \begin{pmatrix}E\mathbb{1} & -\vec\sigma\vec p\\ \vec\sigma\vec p & -E\mathbb{1}\end{pmatrix} + m\mathbb{1}_{4\times4} = \not p + m$$

gdje smo iskoristili,

$$p^\mu \gamma_\mu = p^0\gamma_0 + p^i\gamma_i = E\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + p^i\begin{pmatrix}0 & -\sigma^i\\ \sigma^i & 0\end{pmatrix}$$

– reći nešto o kontekstu. (D. jed. ⇒ zašto kvantizacija? ⇒ uvođenje interakcije ⇒ računamo element prijelaza $\langle 0|\psi_f^\dagger \psi_i|0\rangle$ → udarni presjek, širina raspada)

UDARNI PRESJECI I ŠIRINE RASPADA

Fermijevo zlatno pravilo

– brzina prijelaza iz $i \to f$ je dana

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_i)$$

$$\Gamma_{fi} = \lim_{T\to\infty} \frac{|c_f(T)|^2}{T}$$

$\rho(E_i)$ – gustoća stanja na energiji E_i

T_{fi} – matrični element prijelaza
– u slučaju male perturbacije slobodnog hamiltonijana $H = H_0 + \underbrace{H_i}_{\text{mali}}$ dan perturbativnim razvojem ⇒

$$T_{fi} = \langle f|H_i|i\rangle + \sum_{j\neq i}\frac{\langle f|H_i|j\rangle\langle j|H_i|i\rangle}{E_i - E_j}$$

$|f\rangle$ i $|i\rangle$ su stanja slobodne teorije

– želimo izračunati vjerojatnosti:
1) raspada nestabilne čestice
2) raspršenja čestice (udarni presjek)

– gustoću stanja $\rho(E_i)$ možemo pisati

$\rho(E_i) = \left|\frac{dn}{dE}\right|_{E_i}$; dn broj stanja između E i $E+dE$

$= \int \frac{dn}{dE}\,\delta(E_i - E)\,dE$, pa imamo

$$\Gamma_{fi} = 2\pi\int |T_{fi}|^2 \,\delta(E_i - E)\,dn$$

– iz toga vidimo da nas ubiti zanima gustoća konačnih stanja uz očuvanje energije

– u nerelativističkoj QM ravne valove $\psi(x) = Ae^{i(\vec p\cdot\vec x - Et)}$ normaliziramo u kutiji stranica a

$$\int_0^a dx\int_0^a dy \int_0^a dz\, \psi^*\psi = 1 \Rightarrow A^2 = \frac{1}{a^3} = \frac{1}{V}$$

– normalizacija u kutiji zahtijeva periodičnost valne funkcije $\psi(x+a,y,z) = \psi(x,y,z) \Rightarrow e^{ip_x x} = e^{ip_x(x+a)}$

⇒ $(p_x, p_y, p_z) = (n_x, n_y, n_z)\frac{2\pi}{a}$; n_x, n_y, n_z cijeli brojevi 1, 2, 3

– svako stanje okupira volumen u prostoru momenta

$\Rightarrow d^3p = dp_x\,dp_y\,dp_z = \left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$

– stoga je broj stanja u ljusci $|\vec p|$ do $|\vec p| + dp$

$$dn = 4\pi p^2 dp \cdot \frac{V}{(2\pi)^3} \Rightarrow \rho(E) = \frac{dn}{dE} = \frac{dn}{dp}\underbrace{\left|\frac{dp}{dE}\right|}_{\frac{E}{p}} = \frac{4\pi p^2}{(2\pi)^3}V\cdot\frac{E}{p}$$

- što je sa V?
- V u $\varepsilon(E)$ ulazi $\frac{1}{V}$ iz $T_{fi}=\int \psi_1^* \psi_2^* H_I \psi_a d^3x$ za $a\to 1+2$ $\sim \frac{1}{\sqrt{V}}$ tj $|T_{fi}|^2\sim \frac{1}{V}\cdot V$
- zato je zgodno staviti $V=1$.

$$\Rightarrow dn_i = \frac{d^3p_i}{(2\pi)^3}$$

- za raspad u konačno stanje od N čestica postoji $N-1$ nezavisnih momenata (očuvanje momenta)

$$dn = \prod_{i=1}^{N-1} dn_i = \prod_{i=1}^{N-1}\frac{d^3p_i}{(2\pi)^3} = (2\pi)^3\prod_{i=1}^{N} \frac{d^3p_i}{(2\pi)^3}\,\delta^3\Big(p_a-\sum_{i=1}^{N} p_i\Big)$$

- za $a\to 1+2$

$$\Gamma_{fi} = (2\pi)^4\int |T_{fi}|^2\, \delta(E_a-E_1-E_2)\,\delta^3(\vec p_a-\vec p_1-\vec p_2)\,\frac{d^3\vec p_1}{(2\pi)^3}\,\frac{d^3\vec p_2}{(2\pi)^3}$$

- to je za nerelativističku QM.
- za relativističku moramo valne funkcije normalizirati na $2E$ (Lorentz normalizirane označavamo: ψ')

$$\int_V \psi'^*\psi'\, d^3x = 2E \quad ; \text{ prelazak zahtjeva } \psi\to\psi'=\sqrt{2E}\,\psi$$

- uvodimo invarijantnu amplitudu:

$$M_{fi} = \langle \psi_1'\psi_2'\dots|\hat H_I|\psi_a'\psi_b'\dots\rangle = (2E_1\cdot 2E_2\cdot\dots\cdot 2E_a\, 2E_b\dots)^{\frac{1}{2}}\, T_{fi}$$

$$\Rightarrow \Gamma_{fi} = \frac{(2\pi)^4}{2E_a}\int |M_{fi}|^2\, \delta(E_a-E_1-E_2)\,\delta^3(\vec p_a-\vec p_1-\vec p_2)\,\frac{d^3\vec p_1}{(2\pi)^3 2E_1}\,\frac{d^3\vec p_2}{(2\pi)^3 2E_2}$$

- prvo uočimo da imamo ovisnost $\Gamma\sim \frac{1}{E_a}$ što odgovara dilataciji vremena
- drugo M_{fi} je definiran koristeći Lorentz invarijantno normalizirane valne funkcije, a budući da je $\frac{d^3p}{(2\pi)^3 2E}$ isto Lorentz invarijantno imamo Lorentz invarijantan izraz do na $\frac{1}{E_a}$.

Primjer

Izraz $\int d^4p\,\delta(p^2-m^2)\,\theta(p^0)$ je eksplicitno Lorentz invarijantan

$$\int d^4p\,\delta(p^2-m^2)\,\theta(p^0) = \int d^4p\left(\frac{\delta(p^0-E_{\vec p})}{2E_{\vec p}} + \frac{\delta(p^0+E_{\vec p})}{|-2E_{\vec p}|}\right)\theta(p^0) = \int\frac{d^3p}{2E_{\vec p}}$$

koristimo $\delta(g(x)) = \sum_i \frac{\delta(x-x_i)}{|g'(x_i)|}$, x_i su korijeni tj. nultočke $g(x)$

također,

$$\int \delta(E^2-p^2-m^2)\,dE = \int\left(\frac{\delta(E-E_0)}{2E_0}+\frac{\delta(E+E_0)}{|-2E_0|}\right)dE = \frac{1}{2E_0} \;??$$

1) Raspadi

ako počnemo s N čestica u vremenu δt raspast će ih se

$$\delta N = -N\Gamma\,\delta t \Rightarrow \underline{N(t)=N(0)\,e^{-\Gamma t}}$$

Raspad $a \to 1+2$ u sustavu CM.

$E_a = m_a$

$\vec p_a = 0$

$\vec p_1 = -\vec p_2$

$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 m_a}\int |M_{fi}|^2 \delta(m_a - E_1 - E_2)\,\delta^3(\vec p_1+\vec p_2)\,\frac{d^3p_1}{2E_1}\,\frac{d^3p_2}{2E_2}$$

$$\delta^3(\vec p_1+\vec p_2) \Rightarrow \vec p_1 = -\vec p_2$$

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a}\int |M_{fi}|^2 \frac{1}{4E_1E_2}\,\delta(m_a-E_1-E_2)\, d^3p_1 \quad ; \quad E_2^2 = p_1^2+m_2^2$$

$$d^3\vec p_1 = p_1^2\, dp_1 \sin\theta\, d\theta\, d\phi = p_1^2\, dp_1\, d\Omega$$

$$\Gamma_{fi} = \frac{1}{8\pi^2 m_a}\int |M_{fi}|^2\, \delta\Big(\underbrace{m_a - \sqrt{p_1^2+m_1^2} - \sqrt{p_1^2+m_2^2}}_{f(p_1)}\Big)\, \frac{p_1^2}{4E_1E_2}\, dp_1\, d\Omega$$

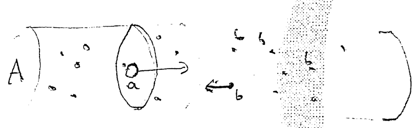
$$\left|\frac{df}{dp_1}\right| = \frac{p_1}{(m_1^2+p_1^2)^{1/2}} + \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = p_1\left(\frac{E_1+E_2}{E_1E_2}\right) \quad ; \quad \text{nultočka } f(p_1 = p^*) = 0$$

$$\Rightarrow p^* = \frac{1}{2m_a}\sqrt{[m_a^2-(m_1+m_2)^2][m_a^2-(m_1-m_2)^2]}$$

$$\Rightarrow \delta(f(p_1)) = \frac{\delta(p_1)}{\left|\frac{df}{dp_1}\right|_{p^*}}$$

$$\Rightarrow \boxed{\Gamma_{fi} = \frac{p^*}{32\pi^2 m_a^2}\int |M_{fi}|^2\, d\Omega}$$

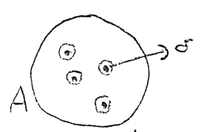
Udarni presjek



- jedna čestica tipa a putuje brzinom $\vec v_a$ kroz volumen određen površinom A
- u tom volumenu nalaze se čestice tipa b s gustoćom n_b koje putuju brzinom $\vec v_b$ u smjeru suprotnom od $\vec v_a$

$\vec v_a, A, n_b, \vec v_b$

- u vremenu δt čestica a preleti volumen koji sadrži $\delta N = n_b(v_a+v_b)A\delta t$ čestica tipa b.



- vjerojatnost sudara je

$$\delta P = \frac{\delta N \sigma}{A} = \frac{n_b(v_a+v_b)A\sigma\delta t}{A} = n_b v \sigma \delta t \quad ; \quad v = v_a+v_b$$

- rata interakcije za svaku česticu tipa a je

stoga $r_a = \frac{dP}{dt} = n_b v \sigma$

- za svaku česticu a, gustoće n_a sadržanu u volumenu V, rata interakcije je

$$R_a = r_a n_a V = (n_b v \sigma) n_a V \underset{\text{preuredimo}}{=} (n_a v)(n_b V)\sigma = \Phi_a N_b \sigma$$

- tj. definicija udarnog presjeka je

$$\sigma = \frac{\text{broj interakcije po jedinici vremena po čestici}}{\text{ulazni tok}}$$

- ako sada normaliziramo valne funkcije na 1 česticu u volumenu $V \Rightarrow n_a = n_b = \frac{1}{V}$
- za ratu interakcije u volumenu V dobijemo

$$\Gamma_{fi} = \frac{v}{V}\sigma$$

- volumen se u konačnom izrazu poništava (za $a+b \to 1+2$)

$|T_{fi}|^2 \sim \frac{1}{V^2}$ $\quad \rho(E) \sim V \Rightarrow \sigma \sim \frac{1}{V^2} V \cdot V \sim 1$

- možemo staviti $V = 1$

$$\Rightarrow \sigma = \frac{\Gamma_{fi}}{v_a + v_b} = \frac{(2\pi)^4}{(v_a+v_b)} \int |T_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}$$

- izrazimo T_{fi} preko Lorentz invarijantne amplitude M_{fi}

$$M_{fi} = (2E_a 2E_b 2E_1 2E_2)^{1/2} T_{fi}$$

$$\sigma = \frac{(2\pi)^{-2}}{4 E_a E_b (v_a + v_b)} \int |M_{fi}|^2 \delta(E_a + E_b - E_1 - E_2)\, \delta^3(\vec{p}_a + \vec{p}_b - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2}$$

- integral je sada Lorentz invarijantan
- veličina $F = 4E_aE_b(v_a + v_b)$ je poznata kao Lorentz invarijantni faktor toka
- Lorentz invarijantnost slijedi iz:

$$F = 4E_aE_b(v_a+v_b) = 4E_aE_b\left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b}\right) = 4(E_a|\vec{p}_b| + E_b|\vec{p}_a|)$$

$$\Rightarrow F^2 = 16(E_a^2|\vec{p}_b|^2 + E_b^2|\vec{p}_a|^2 + 2E_aE_b|\vec{p}_a||\vec{p}_b|)$$

vrijedi samo ako su kolinearni impulsi!!! F nije prava Lorentz invarijanta

- s druge strane:

$$(p_a \cdot p_b)^2 = (E_aE_b + |\vec{p}_a||\vec{p}_b|)^2 = E_a^2E_b^2 + |\vec{p}_a|^2|\vec{p}_b|^2 + 2E_aE_b|\vec{p}_a||\vec{p}_b|$$

- ubacimo $2E_aE_b|\vec{p}_a||\vec{p}_b|$ u F^2

$$\Rightarrow F^2 = 16\left[(p_a\cdot p_b)^2 - (E_a^2 - |\vec{p}_a|^2)(E_b^2 - |\vec{p}_b|^2)\right] \Rightarrow F = 4\left[(p_a\cdot p_b)^2 - m_a^2 m_b^2\right]^{1/2}$$

- pošto su i integral i F Lorentz invarijantni, znači da je i udarni presjek σ Lorentz invarijantan.

- U sustavu CM.

$$\vec{p}_a = -\vec{p}_b = \vec{p}_i^*$$
$$\vec{p}_1 = -\vec{p}_2 = \vec{p}_f^*$$
$$\sqrt{s} = (E_a^* + E_b^*)$$

$$F = 4E_a^*E_b^*(v_a^*+v_b^*) = 4E_a^*E_b^*\left(\frac{p_i^*}{E_a^*}+\frac{p_i^*}{E_b^*}\right) = 4p_i^*(E_a^*+E_b^*) = 4p_i^*\sqrt{s}$$

$$\Rightarrow \quad \sigma = \frac{1}{(2\pi)^2}\frac{1}{4p_i^*\sqrt{s}}\int |M_{fi}|^2\,\delta(\sqrt{s}-E_1-E_2)\,\delta^3(\vec{p}_1+\vec{p}_2)\,\frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}$$

- isti takav se pojavio za raspad $a\to 1+2$ samo je $\sqrt{s}\to m_a$

$$\sigma = \frac{1}{16\pi^2 p_i^*\sqrt{s}}\cdot\frac{p_f^*}{4\sqrt{s}}\int |M_{fi}|^2\,d\Omega^*$$

$d\Omega^*$ - zvjezdica jer se odnosi na SCM (sustav...)

$$\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}\int |M_{fi}|^2\,d\Omega^*$$

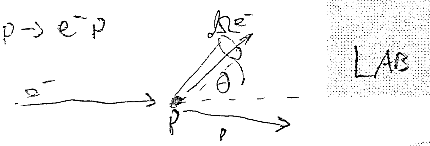
- diferencijalna forma je u SCM (Diferencijalni udarni presjek)

moguće je d... nas zanima: $\frac{d^2\sigma}{d\Omega\,dE}$: sl, ali ne sad

$$d\sigma = \frac{1}{64\pi^2 s}\frac{p_f^*}{p_i^*}|M_{fi}|^2\,d\Omega^*$$

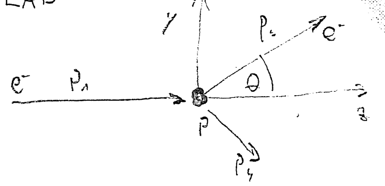
- mjerenja nas može zanimati recimo proces

$e^-p \to e^-p$

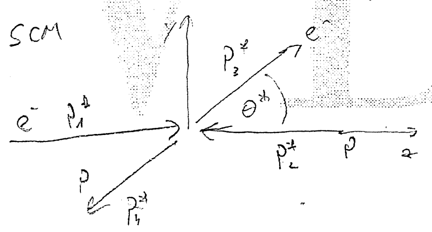


- transformacija diferencijalnog udarnog presjeka u SCM u dif. udarni presjek u LAB sustavu može se provesti tako da prvo izraz za $d\sigma$ napišemo u Lorentz invarijantnoj formi
- želimo riješiti $d\Omega^*$ pomoću Mandelstamove varijable $t=(p_1-p_3)^2$

LAB



SCM



$$t = (p_1^*-p_3^*)^2 = p_1^{*2}+p_3^{*2}-2p_1^*p_3^*$$
$$= m_1^2+m_3^2-2E_1^*E_3^*+2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \quad\Rightarrow\quad dt = 2|\vec{p}_1^*||\vec{p}_3^*|\,d(\cos\theta^*)$$

$$d\Omega^* = d(\cos\theta^*)\,d\phi^* = \frac{dt\,d\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

- identificiramo $|\vec{p}_1^*|\to p_i^*$; $|\vec{p}_3^*|\to p_f^*$ $\Rightarrow$ $d\sigma = \dfrac{1}{128\pi^2 s\, p_i^{*2}}|M_{fi}|^2\,d\phi^*\,dt$

- računamo pretpostavka da je M_{fi} neovisan o ϕ, $d\phi$ daje 2π pod integralom

$$\Rightarrow \frac{d\sigma}{dt} = \frac{1}{64\pi s p_i^{*2}} |M_{fi}|^2$$

- p_i^{*2} je kvadrat iznosa momenta ulaznih čestica u SCM i iznosi

$$p_i^{*2} = \frac{1}{4s}\left[s-(m_1+m_2)^2\right]\left[s-(m_1-m_2)^2\right]$$

- stoga je gornji izraz stvarno Lorentz invarijantan, jer su σ, t, s, i $|M_{fi}|^2$ Lorentz invarijantne
- budući da gornji izraz vrijedi u svakom sustavu, možemo ga iskoristiti i u LAB sustavu

$$p_1 \approx (E_1, 0, 0, E_1) \quad \text{jer } E_1 \gg m_e$$
$$p_2 = (m_p, 0, 0, 0)$$
$$p_3 \approx (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$
$$p_4 = (E_4, \vec{p}_4)$$

$$\Rightarrow p_i^{*2} = \frac{(s-m_p^2)^2}{4s} \quad ; \quad s = (p_1+p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \approx m_p^2 + 2m_p E_1$$

$$\Rightarrow p_i^{*2} = \frac{E_1^2 m_p^2}{m_p^2 + 2E_1 m_p} = \frac{E_1^2}{1 + 2\frac{E_1}{m_p}}$$

- računamo $\frac{d\sigma}{d\Omega}$ — 2π opet od integracije po $d\phi$

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d(\cos\theta)}\frac{1}{2\pi}$$

$$t = (p_1 - p_3)^2 \approx -2E_1E_3(1-\cos\theta)$$

- E_3 je također funkcija od θ

$$t = (p_2 - p_4)^2 = 2m_p^2 - 2p_2 \cdot p_4 = 2m_p^2 - 2m_p E_4 = -2m_p(E_1 - E_3)$$

- zadnji korak je očuvanje energije $E_4 = E_1 + m_p - E_3$

$$\Rightarrow E_3 = \frac{E_1 m_p}{m_p + E_1 - E_1\cos\theta}$$

$$\Rightarrow \frac{dt}{d(\cos\theta)} = 2m_p \frac{dE_3}{d(\cos\theta)} = 2m_p \cdot \frac{E_1^2 m_p}{(m_p + E_1 - E_1\cos\theta)^2} = 2m_p \frac{E_3^2}{m_p} = 2E_3^2$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{2\pi} 2E_3^2 \cdot \frac{d\sigma}{dt} = \frac{E_3^2}{64\pi^2 s p_i^{*2}} |M_{fi}|^2 = \left[\begin{matrix}\text{uvrstimo}\\ p_i^{*2}\end{matrix}\right] = \frac{1}{64\pi^2}\left(\frac{E_3}{m_p E_1}\right)^2 |M_{fi}|^2$$

$$\frac{d\sigma}{d\Omega} = \left[\begin{matrix}\text{uvrstimo}\\ E_3(\theta)\end{matrix}\right] = \frac{1}{64\pi^2}\left(\frac{1}{m_p + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

- isti postupak za $m_e \neq 0$ je algebarski složeniji, ali koraci su isti

ZAOKRUŽUJEMO RASPADE, RASPRŠENJA I REZONANCE

- Raspadi
- izveli smo izraz za Γ_{fi} $(a \to 1+2)$
- Γ_{fi} - po Fermijevom zlatnom pravilu → brzina prijelaza iz $i \to f$ (tj. za jednu česticu $\Gamma_{fi}dt$ je vjerojatnost da će u vremenu dt prijeći iz stanja $i \to f$
- ako počinjemo s $N(t)$ čestica u stanju i u trenutku t, do $t+dt$ će ih u stanje f prijeći

$$dN = -\Gamma N(t)\, dt \Rightarrow N(t) = N(t=0)\, e^{-\Gamma t}$$

- Γ - širina raspada ; $N(t=0) = N_0$

- Zadatak: Izvedite izraz za srednje vrijeme života čestice koja se raspada

$$\tau = \langle t \rangle = \frac{1}{N_0}\int_0^\infty t\, dN(t) = \frac{1}{N_0}\int_0^\infty t\, \Gamma N_0 e^{-\Gamma t} dt = \int_0^\infty dt\, t\, \Gamma e^{-\Gamma t} = \frac{1}{\Gamma}\underbrace{\int_0^\infty d(\Gamma t)\, \Gamma t\, e^{-\Gamma t}}_{=1} = \frac{1}{\Gamma}$$

$$\boxed{\tau = \frac{1}{\Gamma}}$$

Napomena: - ovaj integral je ekvivalentan $\frac{\sum t_i N(t_i)}{\sum N(t_i)}$, gdje je $N(t_i)$ broj čestica koje se raspadne u vremenu t_i do $t_i + \Delta t$

- Zadatak: Odredite vrijeme poluživota čestice koja se raspada

$$\frac{N(t = t_{1/2})}{N(t=0)} = \frac{1}{2} = \frac{N_0 e^{-\Gamma t_{1/2}}}{N_0} \Rightarrow \frac{1}{2} = e^{-\Gamma t_{1/2}} \Rightarrow t_{1/2} = \frac{1}{\Gamma}\ln 2 = \tau \ln 2$$

- Pogledajmo neke raspade čestica

čestica	raspad	τ
μ	$\mu \to e \bar{\nu}_e \nu_\mu$	$2{,}2 \cdot 10^{-6}$ s
τ	$\tau \to \mu \bar{\nu}_\mu \nu_\tau$	$290{,}6 \cdot 10^{-15}$ s
π^0	$\pi^0 \to \gamma\gamma$	$8{,}4 \cdot 10^{-17}$ s
n	$n \to p e^- \bar{\nu}_e$	882 s

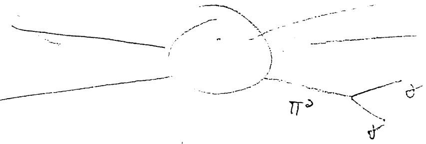
!!! kod visokih energija imamo dilataciju vremena

- kako detektirati ovakve čestice koje se raspadaju?
- npr. $\pi^0 \to \gamma\gamma$ π^0 - živi jako kratko, kako znamo da postoji ako ne ostavlja trag u detektoru?
- očuvanje energije i impulsa nam daje

$$p_{\pi^0} = p_{\gamma_1} + p_{\gamma_2} \quad /^2$$

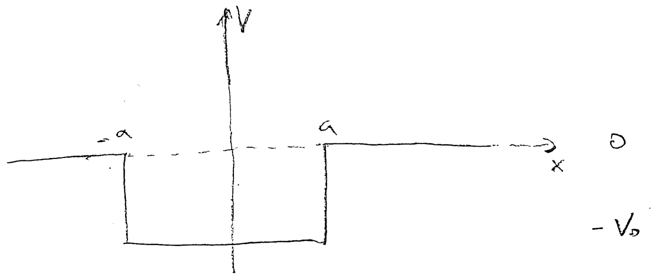
$$m_{\pi^0}^2 = (p_{\gamma_1} + p_{\gamma_2})^2$$

realno imamo puno:



- ako nađemo dva fotona koja zadovoljavaju $(p_{\gamma_1} + p_{\gamma_2})^2 = m_{\pi}^2$; našli smo potpis postojanja π^0

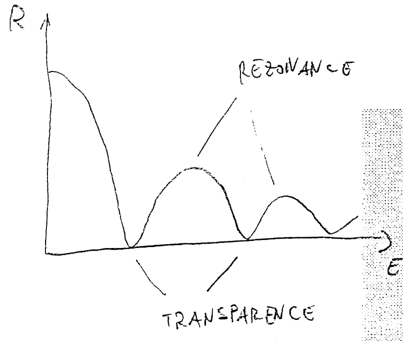
- Rezonance
- gledamo raspršenje na 1D potencijalnoj jami



$$V(x)=\begin{cases}0 & x<-a\\ -V_0 & -a<x<a\\ 0 & x>a\end{cases}$$

$$u_1(x)=e^{ikx}+Re^{-ikx} \qquad k^2=\frac{2mE}{\hbar^2}$$
$$u_2(x)=Ae^{ik'x}+Be^{-ik'x} \qquad k'^2=\frac{2m(E+V_0)}{\hbar^2}$$
$$u_3(x)=Te^{ikx}$$

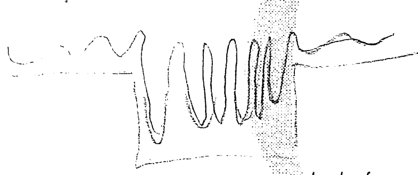
- naštimavanjem rubnih uvjeta u -a i a možemo dobiti R, T, A, B. (R - koeficijent refleksije)



REZONANCE - na tim energijama refleksija je najveća
- najveća interakcija s potencijalom

TRANSPARENCE - na tim energijama nema refleksije
- čestica prolazi preko potencijala kao da ga nema

- rezonance možemo zamisliti kao kvazi vezana stanja → čestica se neko vrijeme zadrži u jami



Napomena! Da se čestica zadrži u jami se ne vidi na valnim funkcijama (valne funkcije su manje u jami nego izvan nje), nego na raspršenju valnih paketa ⇒ (Wigner delay time)

- udarni presjek je mjera međudjelovanja s metom
→ očekujemo minimume i maksimume s obzirom na energiju
→ maksimume, rezonance možemo shvatiti kao kvazi-vezana stanja koja žive neko vrijeme pa se raspadnu → nova čestica

- nova čestica → porast σ
- Breit-Wignerova formula nam opisuje ponašanje udarnog presjeka u blizini rezonance
$1+2\to C^*\to A+B$
- metu-česticu C^* možemo opisati valnom funkcijom

$$\psi(t)=\begin{cases}0 & t<0\\ \psi(0)e^{-iE_0t}e^{-\Gamma t/2}\end{cases} \Rightarrow |\psi(t)|^2=|\psi(0)|^2e^{-\Gamma t}$$ - zadovoljava relaciju raspada čestica

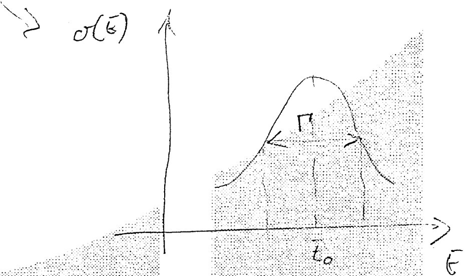
- Fourier-ov transformat valne funkcije

$$\chi(E)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(t)\,e^{iEt}\,dt=\frac{\psi(0)}{\sqrt{2\pi}}\int_{0}^{\infty}e^{-t\left(iE_0-iE+\frac{\Gamma}{2}\right)}\,dt=\frac{\psi(0)}{\sqrt{2\pi}}\,\frac{1}{i(E_0-E)+\frac{\Gamma}{2}}$$

$$\chi(E)=\frac{-i\,\frac{\psi(0)}{\sqrt{2\pi}}}{(E_0-E)-i\frac{\Gamma}{2}}$$

- ne ulazeći u detalje:

$$\sigma(E)\sim\chi^*(E)\,\chi(E)\sim\frac{1}{(E_0-E)^2+\left(\frac{\Gamma}{2}\right)^2}$$ - rezonanca



$$\sigma(E=E_0)\sim\frac{1}{\frac{\Gamma^2}{4}}$$

$$\sigma\left(E=E_0+\frac{\Gamma}{2}\right)\sim\frac{1}{2\,\frac{\Gamma^2}{4}}=\frac{1}{2}\,\sigma(E=E_0)$$

- što metastanje C^* kraće živi, veća je neodređenost u energiji

$\Gamma >> \Rightarrow \tau << \Rightarrow \Delta E >>$

- Dva primjera:

J/ψ - rezonanca (vezano pobuđeno stanje $c\bar{c}$) - pri eksperimentalnom otkriću c-kvarka

<u>ψ eksperiment</u>

- sudaramo e^+e^-

$e^+e^-\to$ hadroni $\Rightarrow$ veliki porast udarnog presjeka na $E=3{,}097\,GeV$

$e^+e^-\to e^+e^-$ $\Rightarrow e^+e^-\to J/\psi\to\dots$

$e^+e^-\to\mu^+\mu^-$

<u>J eksperiment</u>

- sudaramo p i Berilij

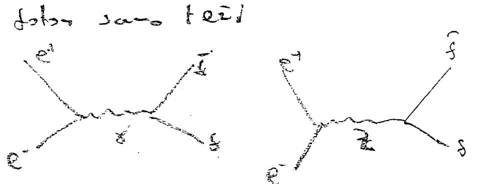
$p+Be\to e^+e^-X$ $\Rightarrow$ porast na $E_{ee}=3{,}097\,GeV$

$\Rightarrow p+Be\to J/\psi\,X\to e^+e^-X$

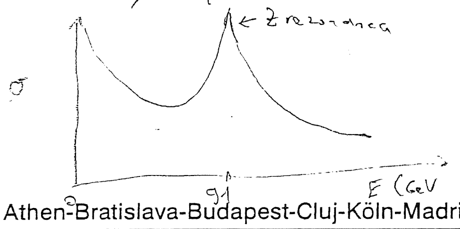
- otkriće Z bozona
- problem je što je isti kao foton samo težak

- npr

$e^+e^-\to f\bar{f}$ može ići



- veliki porast udarnog presjeka na 91,2 GeV $\Rightarrow$ Z bozon



KVANTIZACIJA POLJA (OPERATOR ENERGIJE FERMIONSKOG POLJA)

- Gustoća Lagrangiana za Diracovo polje:

$$\mathcal{L}_D(x) = \bar{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x) \Rightarrow L = \int \mathcal{L}(x)\, d^3x \Rightarrow S = \int L(t)\, dt = \int \mathcal{L}(\bar{\psi}, \psi, \partial_\mu\bar{\psi}, \partial_\mu\psi)\, d^4x$$

- $\delta S = 0$ vodi na Euler-Lagrange jednadžbe

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\right) - \frac{\partial\mathcal{L}}{\partial\bar{\psi}} = 0 \quad i \quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0$$

- Diracova jednadžba slijedi iz prve E-L jednadžbe:

$$-(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$
$$(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

slično druga jednadžba vodi na

$$i\partial_\mu\bar{\psi}(x)\gamma^\mu + m\bar{\psi}(x) = 0$$

- Hamiltonijan možemo dobiti na dva načina:

1' U kl. mehanici

$$H = \sum_i p_i\dot{q}_i - L \quad \text{gdje je} \quad p_i = \frac{\partial L}{\partial\dot{q}_i}$$

- kod nas je

$$p(\vec{x}) = \frac{\partial L}{\partial\dot{\phi}(x)} = \frac{\partial}{\partial\dot{\phi}(x)}\int \mathcal{L}(\phi, \partial_\mu\phi)\, d^3x = \int \frac{\partial}{\partial\dot{\phi}(x)}\mathcal{L}\, d^3x \equiv \int \pi(\vec{x})\, d^3x \quad \text{gdje je} \quad \pi(\vec{x}) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(\vec{x})}$$

tako da za H imamo

$$H = \int d^3x\, [\pi(\vec{x})\dot{\phi}(\vec{x}) - \mathcal{L}] \equiv \int d^3x\, \mathcal{H}$$, gdje je sad $\mathcal{H}$ gustoća Hamiltonijana

2' Iz Noetherinog teorema za invarijantnost na translacije u prostoru i vremenu.

<u>Intermezzo o Noetherinom tm.</u>

- povezuje simetrije klasične teorije polja i zakone sačuvanja
- za svaku kontinuiranu globalnu simetrijsku transformaciju polja postoji jedna sačuvana struja (sačuvani naboj)
- kontinuirana transformacija polja

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha\Delta\phi(x)$$; gdje je α infinitezimalan.

- transformacija je simetrijska ako se njome jednadžbe gibanja.
- to je osigurano ako je akcija invarijantna na transformacije $\phi(x)$ do na površinske članove

$$\mathcal{L}(x) \to \mathcal{L}(x) + \alpha\partial_\mu J^\mu(x) = \mathcal{L}(x) + \alpha\Delta\mathcal{L}(x)$$

$$\alpha\partial_\mu J^\mu(x) = \alpha\Delta\mathcal{L}(x)$$
$$= \frac{\partial\mathcal{L}}{\partial\phi}(\alpha\Delta\phi) + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\mu(\alpha\Delta\phi)\right)$$
$$= \alpha\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi\right) + \alpha\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]}_{=0\ E-L}\Delta\phi$$

$$\Rightarrow \partial_\mu j^\mu(x) \equiv 0$$
$$j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Delta\phi - J^\mu$$
$$\Rightarrow Q = \int_{3D} j^0\, d^3x$$

- za transformaciju koordinata:

$$x^\mu \to x^\mu - a^\mu \Rightarrow \phi(x) \to \phi(x+a) \Rightarrow \phi(x) \xrightarrow{inf} \phi(x) + a^\mu \partial_\mu \phi(x)$$

$$\Rightarrow (\Delta\phi)_\mu = \partial_\mu \phi(x)$$

$$\mathcal{L}(x) \to \mathcal{L}(x+a) \Rightarrow \mathcal{L}(x) \xrightarrow{inf} \mathcal{L}(x) + a^\mu \partial_\mu \mathcal{L}(x) = \mathcal{L}(x) + a^\nu \partial_\mu \underbrace{(\delta^\mu_\nu \mathcal{L}(x))}_{(J^\mu(x))_\nu}$$

- imamo 4 sačuvane struje

$$j^\mu{}_\nu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi(x) - \delta^\mu_\nu \mathcal{L}(x) = T^\mu{}_\nu(x)$$

- tzv. tenzor energije i impulsa
- vrijedi $\partial_\mu T^\mu{}_\nu(x) = 0$

- četiri pridružene sačuvane veličine su

Hamiltonijan:

$$H = \int T^{00} d^3x = \int \mathcal{H} d^3x = \int d^3x \frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi(x) - \mathcal{L}(x) = \int d^3x \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi - \mathcal{L}(x) = \int d^3x\, \pi\dot\phi - \mathcal{L}$$

Impuls:

$$P^i = \int T^{0i} d^3x = \int \frac{\partial\mathcal{L}}{\partial(\dot\phi)} \partial^i\phi(x)\, d^3x = \int \pi(x)\partial^i\phi(x)\, d^3x$$

- kraj intermezza
- Noetherin teorem za translacije u prostoru i vremenu nam daje 4 sačuvane struje iskombinirane u tenzor energije i impulsa:

$$T^\mu{}_\nu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi(x) - \delta^\mu_\nu\mathcal{L}(x)$$

koji zadovoljava jednadžbe kontinuiteta:

$$\partial_\mu T^\mu{}_\nu(x) = 0$$

- a očuvane veličine su

$$H = \int T^{00} d^3x \quad i \quad P^i = \int T^{0i} d^3x$$

- uvrštavajući $T^{00} = \frac{\partial\mathcal{L}}{\partial\dot\phi}\dot\phi(x) - \mathcal{L}(x)$ dobivamo isti izraz kao i prije
- za Diracovo polje gustoća Hamiltonijana je dana izrazom:

$$\mathcal{H}_D = \frac{\partial\mathcal{L}_D}{\partial\dot{\bar\psi}}\dot{\bar\psi}(x) + \frac{\partial\mathcal{L}_D}{\partial\dot\psi}\dot\psi(x) - \mathcal{L}_D(x);$$

- gledajući Diracov Lagrangian imamo:

$$\frac{\partial\mathcal{L}_D}{\partial\dot{\bar\psi}} = 0 \; ; \; \frac{\partial\mathcal{L}_D}{\partial\dot\psi} = \bar\psi i\gamma^0$$

$$\Rightarrow \frac{\partial\mathcal{L}_D}{\partial\dot\psi}\partial_0\psi(x) = \bar\psi i\gamma^0\partial_0\psi$$

$$\Rightarrow \boxed{\mathcal{H}_D = \bar\psi(x)(-i\vec\gamma\vec\nabla + m)\psi(x)}$$ imamo izraz za gustoću hamiltonijana diracovog polja

- do tuda je sve klasična teorija polja
- sada ćemo polje kvantizirati

- Opće rješenje slobodne Diracove jednadžbe (Napravimo Fourierov transformat ili nešto kao valni paket)

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_{s=1,2} \left(a^s_{\vec p} u^s(p) e^{-ipx} + b^{s*}_{\vec p} v^s(p) e^{ipx} \right)$$

$$\bar\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_{s=1,2} \left(a^{+s}_{\vec p} \bar u^s(p) e^{ipx} + b^s_{\vec p} \bar v^s(p) e^{-ipx} \right)$$

- ova polja kvantiziramo.
$\psi, \bar\psi$ - postaju operatori
- ono što su dosad bili koeficijenti
a, b^*, a^*, b sada postaju operatori a, b^+, a^+, b

- s ovim poljima ulazimo u račun energije

$$H = \int d^3x\, \mathcal{H}_D = \int d^3x \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p'}}} \sum_{s'=1,2} \left(a^{+s'}_{\vec p'} \bar u^{s'}(p') e^{ip'x} + b^{s'}_{\vec p'} \bar v^{s'}(p') e^{-ip'x} \right) (-i\vec\gamma\vec\nabla + m) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_{s=1,2} \left(a^s_{\vec p} u^s(p) e^{-ipx} + b^{s+}_{\vec p} v^s(p) e^{ipx} \right)$$

- prvo djelujemo operatorom $(-i\vec\gamma\vec\nabla + m)$

$$(-i\vec\gamma\vec\nabla + m) u^s(p) e^{-ipx} = (-\vec\gamma \cdot \vec p + m) u^s(p) e^{-ipx} = \gamma^0 p_0 u^s(p) e^{-ipx} \quad \text{iz} \quad (\gamma^\mu p_\mu - m) u(p) = 0$$

$$(-i\vec\gamma\vec\nabla + m) v^s(p) e^{ipx} = -\gamma^0 p_0 v(p) e^{ipx} \quad \text{iz} \quad (\gamma^\mu p_\mu + m) v(p) = 0$$

$$H = \int d^3x \int \frac{d^3p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p'}}} \sum_{s'=1,2} \left(a^{+s'}_{\vec p'} \bar u^{s'}(p') e^{ip'x} + b^{s'}_{\vec p'} \bar v^{s'}(p') e^{-ip'x} \right) \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec p}}} \sum_{s=1,2} \left(a^s_{\vec p} \gamma^0 p_0 u^s(p) e^{-ipx} - b^{s+}_{\vec p} \gamma^0 p_0 v^s(p) e^{ipx} \right)$$

- radimo integraciju po $\int \frac{d^3x}{(2\pi)^3}$ i koristimo $\int \frac{d^3x}{(2\pi)^3} e^{i(p-p')x} = \delta(p-p')$

$$H = \int d^3p' \int \frac{d^3p}{(2\pi)^3} \frac{p_0}{\sqrt{2E_{\vec p'}}\sqrt{2E_{\vec p}}} \sum_{ss'} \Big[a^{+s'}_{\vec p'} \underbrace{\bar u^{s'}(p')\gamma^0}_{u^{+s'}(p')} a^s_{\vec p} u^s(p) \delta(p-p') + b^{s'}_{\vec p'} v^{+s'}(p') a^s_{\vec p} u^s(p) \delta(p+p') - a^{+s'}_{\vec p'} u^{+s'}(p) b^{+s}_{\vec p} v^s(p) \delta(p+p')$$
$$- b^{s'}_{\vec p'} v^{+s'}(p') b^{+s}_{\vec p} v^s(p) \delta(p'-p) \Big]$$

- koristimo prije izvedene relacije
$u^{r+}(p) u^s(p) = 2E_{\vec p} \delta^{rs}$
$v^{r+}(p) v^s(p) = 2E_{\vec p} \delta^{rs}$
$u^{r+}(p) v^s(-p) = v^{r+}(p) u^s(-p) = 0$
- i provodimo integral po p'

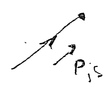
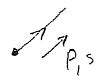
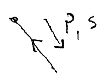
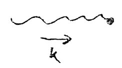
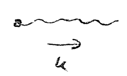
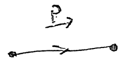
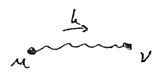
$$H = \int \frac{d^3p}{(2\pi)^3} \frac{E_{\vec p}}{2E_{\vec p}} \sum_s 2E_{\vec p} \left(a^{+s}_{\vec p} a^s_{\vec p} - b^s_{\vec p} b^{+s}_{\vec p} \right) = \int \frac{d^3p}{(2\pi)^3} E_{\vec p} \sum_s \left(a^{+s}_{\vec p} a^s_{\vec p} - b^s_{\vec p} b^{+s}_{\vec p} \right)$$

- problem s ovim operatorom energije je što nije pozitivno definitan. Ima član sa -
- kako to riješiti
- operatori broja čestica (antičestica) su $N = a^+ a \ (b^+ b)$
- inače u QM kvantiziramo komutatorima $[x, p] = i\hbar$, $[x, y] = 0$
- ako ovdje probamo s $[b, b^+] = \delta^3(\vec p' - \vec p)\delta^{rs}$ opet dobijemo -
- moramo kvantizirati antikomutatorima!!! $\{b, b^+\} = \delta^3(\vec p' - \vec p)\delta^{rs}$, $\{b^+, b\} = 0$ i svi ostali 0

$b^{+s}_{\vec p} b^{+s}_{\vec p} |0\rangle = -b^{+s}_{\vec p} b^{+s}_{\vec p} |0\rangle = 0$

- Fermi-Diracova statistika!!!
...

FEYNMANOVA PRAVILA ZA QED

Komponenta dijagrama	Opis	Oznaka (faktor)
	elektron u početnom stanju (ulazni elektron)	$u^s(p)$
	izlazni elektron	$\bar{u}^s(p)$
	ulazni pozitron	$\bar{v}^s(p)$
	izlazni pozitron	$v^s(p)$
	ulazni foton	$\varepsilon^\mu(k,\lambda)$
	izlazni foton	$\varepsilon^{\mu *}(k,\lambda)$
	elektronski propagator (virtualni elektron)	$i\dfrac{\not p + m}{p^2-m^2}$
	fotonski propagator	$\dfrac{-i g_{\mu\nu}}{k^2}$
	vrh QED-a	$-ie\gamma^\mu$

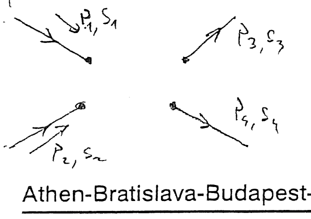
- očuvanje impulsa u vrhu
- dijagrami koji se međusobno razlikuju samo za izmjenu identičnih vanjskih fermionskih linija dobivaju faktor (-1) pri računanju ukupne amplitude

Zadatak: Promotrite Møllerovo raspršenje. $e^-e^- \to e^-e^-$. Nacrtajte sve Feynmanove dijagrame u 2. redu računa smetnje. Napišite izraz za invarijantnu amplitudu M_{fi}. Izračunajte usrednjeni, nepolarizirani kvadrat amplitude $\overline{|M|^2}$ i izrazite rezultat pomoću Mandelstamovih varijabli. Izrazite diferencijalni udarni presjek u SCM.

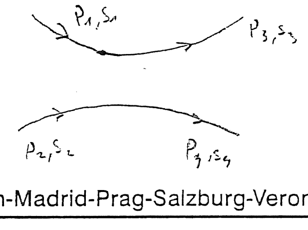
Proces. $(e^-e^- \to e^-e^-)$

Iz Feynmanovih pravila:

- kako možemo pospajati vanjske linije



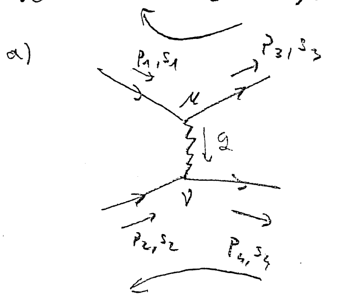
O. red (nezanima nas)



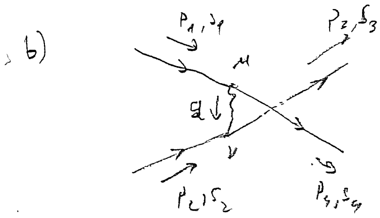
- nema interakcije $\langle f|i\rangle = \delta_{fi}$
- ne ulazi u M_{fi}

2. red računa smetnje

a)



b)



$$-iM_a = \bar{u}^{s_3}(p_3)(-ie\gamma^\mu)u^{s_1}(p_1)\,\bar{u}^{s_4}(p_4)(-ie\gamma^\nu)u^{s_2}(p_2)\,\frac{-ig_{\mu\nu}}{q^2}$$

- zakon sačuvanja četveroimpulsa $q = p_1 - p_3$
- t; $q^2 = (p_1-p_3)^2 = t$

$$-iM_b = \bar{u}_4(-ie\gamma^\mu)u_1\,\bar{u}_3(-ie\gamma^\nu)u_2\,\frac{-ig_{\mu\nu}}{q^2} \qquad q^2 = (p_1-p_4)^2 = u$$

- ukupna amplituda
- obično $M_{uk} = M_a + M_b$, međutim a i b se razlikuju samo do na zamjenu $3 \leftrightarrow 4$, pa nam zadnje Feynmanovo pravilo daje $M_{uk} = M_a - M_b$

$$M_{uk} = -\frac{e^2}{t}\bar{u}_3\gamma^\mu u_1\bar{u}_4\gamma_\mu u_2 - \left(-\frac{e^2}{u}\bar{u}_4\gamma^\mu u_1\bar{u}_3\gamma_\mu u_2\right)$$

- da bi izračunali kvadrat amplitude treba nam $M_{uk}^\dagger$ ($|M|^2 = M^\dagger M$)

$$M^\dagger = -\frac{e^2}{t}[\bar{u}_3\gamma^\mu u_1]^\dagger[\bar{u}_4\gamma_\mu u_2]^\dagger + \frac{e^2}{u}[\bar{u}_4\gamma^\mu u_1]^\dagger[\bar{u}_3\gamma_\mu u_2]^\dagger =$$

$$= -\frac{e^2}{t}[u_1^\dagger\gamma^{\mu\dagger}\bar{u}_3^\dagger\, u_2^\dagger\gamma_\mu^\dagger\bar{u}_4^\dagger] + \frac{e^2}{u}[u_1^\dagger\gamma^{\mu\dagger}\bar{u}_4^\dagger\, u_2^\dagger\gamma_\mu^\dagger\bar{u}_3^\dagger] =$$

$$= -\frac{e^2}{t}[u_1^\dagger\gamma^0\gamma^\mu\underbrace{\gamma^0\gamma^0}_{1}u_3\, u_2^\dagger\gamma^0\gamma_\mu\gamma^0\gamma^0u_4] + \frac{e^2}{u}[u_1^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^0u_4\, u_2^\dagger\gamma^0\gamma_\mu\gamma^0\gamma^0u_3]$$

$$\boxed{\begin{array}{l}\text{koristimo:}\\ \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0\\ \bar{u}^\dagger = [u^\dagger\gamma^0]^\dagger = [\gamma^{0\dagger}u] = \gamma^0 u\\ \gamma^0\gamma^0 = 1\end{array}}$$

$$M^\dagger = -\frac{e^2}{t}[\bar{u}_1\gamma^\mu u_3\bar{u}_2\gamma_\mu u_4] + \frac{e^2}{u}[\bar{u}_1\gamma^\mu u_4\bar{u}_2\gamma_\mu u_3]$$

$$|M|^2 = M^\dagger M = \frac{e^4}{t^2}[\bar{u}_1\gamma^\mu u_3\bar{u}_2\gamma_\mu u_4\bar{u}_3\gamma^\nu u_1\bar{u}_4\gamma_\nu u_2] - \frac{e^4}{ut}[\bar{u}_1\gamma^\mu u_4\bar{u}_2\gamma_\mu u_3\bar{u}_3\gamma^\nu u_1\bar{u}_4\gamma_\nu u_2] - \frac{e^4}{ut}[\bar{u}_1\gamma^\mu u_3\bar{u}_2\gamma_\mu u_4\bar{u}_4\gamma^\nu u_1\bar{u}_3\gamma_\nu u_2]$$

$$+ \frac{e^4}{u^2}[\bar{u}_1\gamma^\mu u_4\bar{u}_2\gamma_\mu u_3\bar{u}_4\gamma^\nu u_1\bar{u}_3\gamma_\nu u_2]$$

- nas zanima usrednjeni nepolarizirani kvadrat amplitude (ne zanimaju nas spinovi izlaznih čestica, a imamo i nepolarizirani ulazni snop)
- usrednjavamo po ulaznim spinovima i zbrajamo kvadrate amplitudi za izlazne spinove

$$\overline{|M|^2} = \underbrace{\frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)}\sum_{s_1,s_2}}_{\text{usrednjavanje po ulaznim spinovima}}\underbrace{\sum_{s_3,s_4}}_{\text{zbroj po izlaznim}}|M|^2$$

- za izračun sumi koristimo Casimirov trik
- pogledajmo prvi član u $|M|^2$

$$\bar{u}_1\gamma^\mu u_3\bar{u}_2\gamma_\mu u_4\bar{u}_3\gamma^\nu u_1\bar{u}_4\gamma_\nu u_2 = \bar{u}_1\gamma^\mu u_3\bar{u}_3\gamma^\nu u_1 \cdot \bar{u}_2\gamma_\mu u_4\bar{u}_4\gamma_\nu u_2$$

- pogledajmo svaki od ovih za sebe

1) $\bar{u}_1\gamma^\mu u_3\bar{u}_3\gamma^\nu u_1 = (\bar{u}_1)_\alpha(\gamma^\mu)_{\alpha\beta}(u_3)_\beta(\bar{u}_3)_\gamma(\gamma^\nu)_{\gamma\delta}(u_1)_\delta = (u_1)_\delta(\bar{u}_1)_\alpha(\gamma^\mu)_{\alpha\beta}(u_3)_\beta(\bar{u}_3)_\gamma(\gamma^\nu)_{\gamma\delta} =$

- raspišemo po komponentama
- ovo su sad brojevi (pa prebacimo zadnji na početak)

$$= (u_1\bar{u}_1)_{\delta\alpha}(\gamma^\mu)_{\alpha\beta}(u_3\bar{u}_3)_{\beta\gamma}(\gamma^\nu)_{\gamma\delta} = Tr[u_1\bar{u}_1\gamma^\mu u_3\bar{u}_3\gamma^\nu]$$

- iskombiniramo $u_i\bar{u}_i$ u matricu
- ispadne nam van trag
- iskoristimo sada prije izvedenu relaciju potpunosti:

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \not{p}+m$$

- u ultrarelativističkom limesu (URL) $m \to 0$

$$\Rightarrow \sum_{s_1,s_3} Tr[u_1\bar{u}_1\gamma^\mu u_3\bar{u}_3\gamma^\nu] = Tr[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu] = 4(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - (p_1\cdot p_3)g^{\mu\nu})$$

2) $\bar{u}_2\gamma_\mu u_4\bar{u}_4\gamma_\nu u_2 = (\bar{u}_2)_\alpha(\gamma_\mu)_{\alpha\beta}(u_4)_\beta(\bar{u}_4)_\gamma(\gamma_\nu)_{\gamma\delta}(u_2)_\delta = (u_2\bar{u}_2)_{\delta\alpha}(\gamma_\mu)_{\alpha\beta}(u_4\bar{u}_4)_{\beta\gamma}(\gamma_\nu)_{\gamma\delta} = Tr[u_2\bar{u}_2\gamma_\mu u_4\bar{u}_4\gamma_\nu]$

$$\Rightarrow \sum_{s_2,s_4} Tr[u_2\bar{u}_2\gamma_\mu u_4\bar{u}_4\gamma_\nu] = Tr[\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu] = 4(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - (p_2\cdot p_4)g_{\mu\nu})$$

$\Rightarrow$ prvi član u $\overline{|M|^2}$ je

$$\frac{e^4}{t^2}\underbrace{\frac{1}{(2s_1+1)}\frac{1}{(2s_2+1)}}_{\frac{1}{4}}32[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)] = \frac{e^4}{u^2}\cdot 8[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)]$$

- tu smo koristili $Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2]Tr[\gamma_\mu\not{p}_3\gamma_\nu\not{p}_4] = 32[(p_1\cdot p_3)(p_2\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)]$

- na sličan način se izračunaju ostala tri člana u $\overline{|M|^2}$:
- zadnji član u $\overline{|M|^2}$ npr.

$$\sum_{s_1 s_2 s_3 s_4}\bar{u}_1\gamma^\mu u_4\bar{u}_4\gamma_\nu u_2\bar{u}_2\gamma_\mu u_3\bar{u}_3\gamma^\nu u_1 = \sum_s Tr[u_1\bar{u}_1\gamma^\mu u_4\bar{u}_4\gamma_\nu u_2\bar{u}_2\gamma_\mu u_3\bar{u}_3\gamma^\nu] = Tr[\not{p}_1\gamma^\mu\not{p}_4\gamma_\nu\not{p}_2\gamma_\mu\not{p}_3\gamma^\nu] = -32(p_1\cdot p_2)(p_3\cdot p_4)$$

- tu smo koristili $Tr[\gamma^\mu\not{p}_1\gamma^\nu\not{p}_2\gamma_\mu\not{p}_3\gamma_\nu\not{p}_4] = -32(p_1\cdot p_3)(p_2\cdot p_4)$

- ukupni usrednjeni kvadrat amplitude je

$$\overline{|M|^2} = \frac{e^4}{4}\Big[\frac{1}{t^2}Tr[\not{p}_1\gamma^\mu\not{p}_3\gamma^\nu]Tr[\not{p}_2\gamma_\mu\not{p}_4\gamma_\nu] + \frac{1}{u^2}Tr[\not{p}_1\gamma^\mu\not{p}_4\gamma^\nu]Tr[\not{p}_2\gamma_\mu\not{p}_3\gamma_\nu] - \frac{1}{ut}Tr[\not{p}_1\gamma^\mu\not{p}_4\gamma_\nu\not{p}_2\gamma_\mu\not{p}_3\gamma^\nu]$$

$$-\frac{1}{ut}Tr[\not{p}_1\gamma^\mu\not{p}_3\gamma_\nu\not{p}_2\gamma_\mu\not{p}_4\gamma^\nu]\Big] = \frac{e^4}{4}\Big[\frac{32}{t^2}[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)] + \frac{32}{u^2}[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_3)(p_2\cdot p_4)] +$$

$$+\frac{32}{ut}[2(p_1\cdot p_2)(p_3\cdot p_4)]\Big]$$

- ovaj rezultat treba izraziti preko Mandelstamovih varijabli:

$$s = (p_1+p_2)^2 = m_1^2+m_2^2+2p_1\cdot p_2 \Rightarrow p_1\cdot p_2 = p_3\cdot p_4 = \frac{s}{2}$$

$$t \approx -2p_1\cdot p_3 = -2p_2\cdot p_4 \Rightarrow p_1\cdot p_3 = p_2\cdot p_4 = -\frac{t}{2}$$

$$u \approx -2p_1\cdot p_4 = -2p_2\cdot p_3 \Rightarrow p_1\cdot p_4 = p_2\cdot p_3 = -\frac{u}{2}$$

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4}\left[\frac{32}{t^2}\left[\frac{s}{2}\cdot\frac{s}{2} + \left(-\frac{u}{2}\right)\left(-\frac{u}{2}\right)\right] + \frac{32}{u^2}\left[\frac{s}{2}\frac{s}{2} + \left(-\frac{t}{2}\right)\left(-\frac{t}{2}\right)\right] + \frac{32}{ut}\left[2\cdot\frac{s}{2}\frac{s}{2}\right]\right] =$$

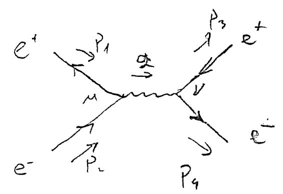
$$= \frac{e^4}{4}\frac{32}{4t^2u^2}\left[u^4+t^4+s^2\underbrace{(u+t)^2}_{s+t+u=\underbrace{4m^2}_{0 \text{ u URL}}}\right] = \boxed{\frac{2e^4}{t^2u^2}\left[u^4+t^4+s^4\right]}$$

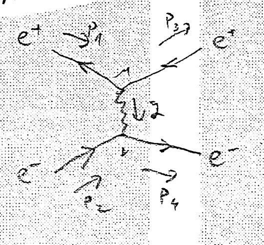
- diferencijalni udarni presjek

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\overline{|\mathcal{M}|^2} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\frac{2e^4}{t^2u^2}\left[s^4+t^4+u^4\right]\cdot\frac{1}{2}\ !!!$$

- faktor 1/2 jer imamo identične čestice u konačnom stanju
- u izrazu za amplitudu je već uključeno da su izlazne čestice neraspoznatljive
- ne znamo koja čestica se raspršila u θ, pa je dovoljno samo pola prostornog kuta za cijeli udarni presjek
- alternativno $\frac{d\sigma}{d\Omega}$ može ići bez $\frac{1}{2}$, ali onda integriramo samo preko pola prostornog kuta

- <u>križna simetrija</u>
- promotrimo Bhabhino raspršenje $e^- + e^+ \to e^- + e^+$

b)


a)


- a) dijagram je isti kao dijagram a) za Møllerovo raspršenje. Osim što
  $\bar{u}_3\gamma^\mu u_1 \to \bar{v}_1\gamma^\mu v_3$, pa Casimirov trik daje $\sum \bar{v}_1 v_1 = (\not{p}_1 - m)$ zanemarivo u URL

- b) dijagram se od dijagrama b) za Møllerovo raspršenje razlikuje po
  $\bar{u}_4\gamma^\mu u_1 \to \bar{v}_1\gamma^\mu u_2$
  $\bar{u}_3\gamma^\mu u_2 \to \bar{u}_4\gamma^\mu v_3$
  što opet ne daje razliku u Casimirovom triku ali imamo
  $p_4 \to -p_1$
  $p_1 \to p_2$
  $p_3 \to p_4$
  $p_2 \to -p_3$
  tj. $u = (p_1 - p_3)^2 \to (p_2 + p_1)^2 = s$

  - sugerira treba zamijeniti u sa s

- dif. udarni presjek za Bhabhino raspršenje

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2 s}\frac{p_f}{p_i}\frac{2e^4}{t^2s^2}\left[s^4+t^4+u^4\right]$$

- tu nema faktora $\frac{1}{2}$ sad (nemamo iste čestice u izlaznom kanalu).

- izrazimo diferencijalni udarni presjek pomoću kuta raspršenja u SCM: $\left(\begin{array}{l} E_1=E_2=E_3=E_4 \\ |\vec p_1|=|\vec p_2|=|\vec p_3|=|\vec p_4| \end{array}\right.$

$$s=(p_1+p_2)^2 = 4E^2 = 4(|\vec p|^2+m^2)$$

$$t=(p_1-p_3)^2 = \underbrace{(E_1-E_3)^2}_{0} - (\vec p_1-\vec p_3)^2 = 0 - |\vec p_1|^2 - |\vec p_3|^2 + 2|\vec p_1||\vec p_3|\cos\theta = -2|\vec p|^2(1-\cos\theta)$$

$$u=(p_2-p_3)^2 = -2|\vec p|^2(1-\cos(\pi-\theta)) = -2|\vec p|^2(1+\cos\theta)$$

- diferencijalni udarni presjek za Møllerovo raspršenje:

$$\left(\frac{d\sigma}{d\Omega}\right)_{CM} = \frac{1}{64\pi^2}\frac{e^4}{4E^2}\cdot\frac{1}{16p^8(1-\cos\theta)^2(1+\cos\theta)^2}\cdot\left[256E^4 + 16|\vec p|^8(1-\cos\theta)^4 + 16|\vec p|^8(1+\cos\theta)^4\right]$$

$$= \frac{1}{64\pi^2}\frac{e^4}{4E^2}\frac{1}{16p^8\,(1-\cos^2\theta)^2}\left[256p^8 + 16p^8\left((1-\cos\theta)^4+(1+\cos\theta)^4\right)\right] =$$

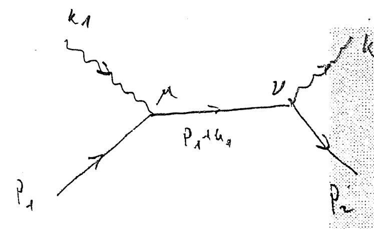
$$= \frac{1}{64\pi^2}\frac{e^4}{4E^2}\cdot\frac{1}{\sin^4\theta}\cdot\left[(3+\cos^2\theta)^2+3^2\right]$$

Comptonovo raspršenje

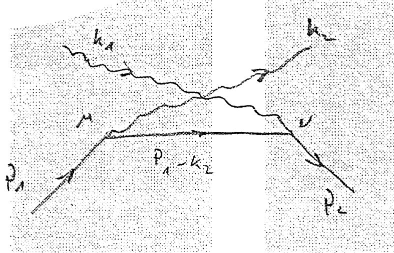
$\gamma e^- \to \gamma e^-$

- u vodećem redu računa smetnje imamo dva dijagrama:

a)



b)



$M_{fi} = M_a + M_b$; - opet koristimo skraćeni zapis $\bar u(p_2,s_2)=\bar u_2$

$$-iM_a = \bar u_2(ie\gamma^\nu)\frac{i(\not p_1+\not k_1+m)}{(p_1+k_1)^2-m^2}(ie\gamma^\mu)u_1\,\varepsilon_\mu(k_1,\lambda_1)\varepsilon^*_\nu(k_2,\lambda_2)$$

$$-iM_b = \bar u_2(ie\gamma^\nu)\frac{i(\not p_1-\not k_2+m)}{(p_1-k_2)^2-m^2}(ie\gamma^\mu)u_1\,\varepsilon_\nu(k_1,\lambda_1)\varepsilon^*_\mu(k_2,\lambda_2)$$

$$M_{fi} = e^2\bar u_2\left[\gamma^\nu\frac{(\not p_1+\not k_1+m)}{(p_1+k_1)^2-m^2}\gamma^\mu + \gamma^\mu\frac{\not p_1-\not k_2+m}{(p_1-k_2)^2-m^2}\gamma^\nu\right]u_1\,\varepsilon_\mu(k_1,\lambda_1)\varepsilon^*_\nu(k_2,\lambda_2)$$

$$|M_{fi}|^2 = M_{fi}^\dagger M_{fi} = e^4\left\{\bar u_2\left[\gamma^\alpha\frac{(\not p_1+\not k_1+m)}{(p_1+k_1)^2-m^2}\gamma^\beta + \gamma^\beta\frac{\not p_1-\not k_2+m}{(p_1-k_2)^2-m^2}\gamma^\alpha\right]u_1\right\}^\dagger\varepsilon^*_\beta(k_1,\lambda_1)\varepsilon_\alpha(k_2,\lambda_2)\times$$

$$\times\,\varepsilon_\mu(k_1,\lambda_1)\varepsilon^*_\nu(k_2,\lambda_2)\left\{\bar u_2\left[\underbrace{\gamma^\nu\frac{\not p_1+\not k_1+m}{(p_1+k_1)^2-m^2}\gamma^\mu+\gamma^\mu\frac{\not p_1-\not k_2+m}{(p_1-k_2)^2-m^2}\gamma^\nu}_{V^{\nu\mu}}\right]u_1\right\}$$

- uvedimo pokratu

$$V^{\alpha\beta} = \left[\gamma^\alpha\frac{(\not p_1+\not k_1+m)}{(p_1+k_1)^2-m^2}\gamma^\beta + \gamma^\beta\frac{\not p_1-\not k_2+m}{(p_1-k_2)^2-m^2}\gamma^\alpha\right]$$

$$\Rightarrow |M_{fi}|^2 = e^4\varepsilon^*_\beta(k_1,\lambda_1)\varepsilon_\alpha(k_2,\lambda_2)\varepsilon_\mu(k_1,\lambda_1)\varepsilon^*_\nu(k_2,\lambda_2)\{\bar u_2V^{\alpha\beta}u_1\}^\dagger\{\bar u_2V^{\nu\mu}u_1\}$$

- za $\{\bar u_2 V^{\alpha\beta}u_1\}^\dagger = \underbrace{u_1^\dagger}_{\gamma^0\gamma^0}V^{\alpha\beta\dagger}(u_2^\dagger\gamma^0)^\dagger = \bar u_1\gamma^0V^{\alpha\beta\dagger}\gamma^0u_2 = \bar u_1\underbrace{\gamma^0V^{\alpha\beta\dagger}\gamma^0}_{\overline{V^{\alpha\beta}}}u_2 = \bar u_1\,\overline{V^{\alpha\beta}}\,u_2$

$\overline{V^{\alpha\beta}}$ - uobičajeno u literaturi se koristi ova oznaka

- koristimo:

$$\overline{\not a_1\not a_2\cdots\not a_n} = \gamma^0(\not a_1\not a_2\cdots\not a_n)^\dagger\gamma^0 = \gamma^0\not a_n^\dagger\gamma^0\gamma^0\cdots\gamma^0\gamma^0\not a_1^\dagger\gamma^0 = \not a_n\cdots\not a_2\not a_1$$

$$\Rightarrow \overline{V^{\alpha\beta}} = V^{\beta\alpha}$$

- nadalje koristimo Casimirov trik:

$$\bar u_1 V^{\beta\alpha} u_2 \bar u_2 V^{\nu\mu} u_1 = (\bar u_1)_\sigma (V^{\beta\alpha})_{\sigma\varrho}(u_2)_\varrho (\bar u_2)_{\sigma'} (V^{\nu\mu})_{\sigma'\varrho'} (u_1)_{\varrho'} = Tr[u_1\bar u_1 V^{\beta\alpha} u_2 \bar u_2 V^{\nu\mu}]$$

$\Rightarrow$

- za kvadrat amplitude sada imamo:

$$|M_{fi}|^2 = e^4\, \varepsilon^*_\beta(k_1,\lambda_1)\varepsilon_\alpha(k_2,\lambda_2)\varepsilon_\mu(k_1,\lambda_1)\varepsilon^*_\nu(k_2,\lambda_2)\, Tr[u_1\bar u_1 V^{\beta\alpha} u_2\bar u_2 V^{\nu\mu}]$$

- usrednjena amplituda je

$$\overline{|M_{fi}|^2} = \frac14 \sum_{s_1,s_2,\lambda_1,\lambda_2} |M_{fi}|^2$$

- koristimo $\sum_s u(p,s)\bar u(p,s) = \not p + m$

$\sum_\lambda \varepsilon_\mu(k,\lambda)\varepsilon^*_\nu(k,\lambda) = -g_{\mu\nu}$ - Thomson MPP - Appendix D

$$\overline{|M_{fi}|^2} = \frac{e^4}{4}(-g_{\mu\beta})(-g_{\alpha\nu})\, Tr[(\not p_1+m)V^{\beta\alpha}(\not p_2+m)V^{\nu\mu}] = \frac{e^4}{4} Tr[(\not p_1+m)V^{\beta\alpha}(\not p_2+m)V_{\alpha\beta}]$$

- nadalje da pojednostavnimo stvari radimo u URL ($m\to0$)

$$\overline{|M_{fi}|^2} = \frac{e^4}{4} Tr\left[\not p_1\left(\gamma^\beta \frac{\not p_1+\not k_1}{s}\gamma^\alpha + \gamma^\alpha\frac{\not p_1-\not k_2}{u}\gamma^\beta\right)\not p_2\left(\gamma_\alpha\frac{\not p_1+\not k_1}{s}\gamma_\beta + \gamma_\beta\frac{\not p_1-\not k_2}{u}\gamma_\alpha\right)\right]$$

$$= \frac{e^4}{4}\cdot\Big[\frac{1}{s^2} Tr[\not p_1 \gamma^\beta (\not p_1+\not k_1)\gamma^\alpha \not p_2 \gamma_\alpha (\not p_1+\not k_1)\gamma_\beta] + \quad (1)$$

$$+ \frac{1}{u^2} Tr[\not p_1 \gamma^\alpha (\not p_1-\not k_2)\gamma^\beta \not p_2 \gamma_\beta (\not p_1-\not k_2)\gamma_\alpha] + \quad (2)$$

$$+ \frac{1}{su} Tr[\not p_1 \gamma^\beta (\not p_1+\not k_1)\gamma^\alpha \not p_2 \gamma_\beta (\not p_1-\not k_2)\gamma_\alpha] + \quad (3)$$

$$+ \frac{1}{su} Tr[\not p_1 \gamma^\alpha (\not p_1-\not k_2)\gamma^\beta \not p_2 \gamma_\alpha (\not p_1+\not k_1)\gamma_\beta]\Big] \quad (4)$$

- 2 = 1 uz $k_1\to -k_2$
- 4 = 3 uz $k_1 \to -k_2$
  $k_2\to -k_1$

(1) - iskoristimo $\gamma_\mu \not p \gamma^\mu = -2\not p$

$$Tr[-2\not p_1(\not p_1+\not k_1)(-2\not p_2)(\not p_1+\not k_1)] = *$$

- iskoristimo $Tr[\not a\not b\not c\not d] = 4((ab)(cd) - (ac)(bd) + (ad)(bc))$

$$* = 16\{2(p_1\cdot(p_1+k_1))(p_2\cdot(p_1+k_1)) - p_1\cdot p_2 (p_1+k_1)^2\} = 16\{2\cdot(\overset{\to0}{\cancel{m^2}}+\tfrac s2)(-\tfrac t2 - \tfrac u2) - (-\tfrac t2)(\overset{\to0}{m^2}+2\tfrac s2+0)\} =$$

$$= 16\{\cancel{2(\tfrac s2)(-\tfrac t2)}+2(\tfrac s2)(-\tfrac u2)\cancel{-2(-\tfrac t2)(\tfrac s2)}\} = \underline{-8su}$$

- koristimo

$p_1\cdot k_1 = p_2\cdot k_2 = \frac s2$

$p_1\cdot k_2 = p_2\cdot k_1 = -\frac u2$

$p_1\cdot p_2 = k_1\cdot k_2 = -\frac t2$

$$\Rightarrow (2) = 16\{2(p_1\cdot(p_1-k_2))(p_2\cdot(p_1-k_2)) - p_1\cdot p_2(p_1-k_2)^2\} = 16\{2(\tfrac u2)(-\tfrac t2-\tfrac s2) - (\tfrac t2)(-2(-\tfrac u2))\}$$

$$= \underline{-8su}$$

3\. $= Tr[\not{p}_1 \gamma^\beta (\not{p}_1 + \not{k}_1) \gamma^\alpha \not{p}_2 \gamma_\beta (\not{p}_1 - \not{k}_2) \gamma_\alpha] = *$

\- možemo iskoristiti:

$\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma_\mu = -2\gamma^\sigma \gamma^\rho \gamma^\nu$; : $\gamma^\mu \gamma^\nu \gamma^\rho \gamma_\mu = 4 g^{\nu\rho} \mathbb{1} \Rightarrow \gamma^\mu \not{a} \not{b} \gamma_\mu = 4 g^{\nu\rho} \mathbb{1} a_\nu b_\rho = 4 a \cdot b$

$* = Tr[\not{p}_1 (-2) \not{p}_2 \gamma^\alpha (\not{p}_1 + \not{k}_1)(\not{p}_1 - \not{k}_2) \gamma_\alpha] = -8 Tr[\not{p}_1 \not{p}_2 (p_1 + k_1)(p_1 - k_2)] = -8 Tr[\not{p}_1 \not{p}_2](p_1 + k_1)(p_1 - k_2) = -32\, p_1 \cdot p_2 (p_1 + k_1)(p_1 - k_2)$

$= -32 \left[\left(-\frac{t}{2}\right) \left(m^2 + \frac{u}{2} + \frac{s}{2} + \frac{t}{2} \right) \right] = -32 \left[\left(-\frac{t}{2}\right) \cdot 3m^2 \right] \simeq 0$

$s + t + u = 4m^2$

5\. $\Rightarrow$ analogno $= 0$

\- konačni izraz za $\overline{|M_{fi}|}^2$

$\overline{|M_{fi}|}^2 = 2e^4 \left(-\frac{u}{s} - \frac{s}{u} \right)$

\- sada se križnom simetrijom može dobiti i izraz za proces $e^+e^- \to \gamma\gamma$

Ponavljanje za kolokvij

1.) a) Snop protona udara u vodikovu metu. Odredite minimalnu energiju snopa da može doći do reakcije:

$$pp \longrightarrow pp\pi^0$$

b) Odredite impuls π^0 mezona u lab sustavu i kut između ulaznog snopa i impulsa π^0 mezona u tom slučaju!

$\underset{1}{p} + \underset{2}{p} \to \underset{3}{p} + \underset{4}{p} + \underset{5}{\pi^0}$

a) u SCM

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 = 4E^2$$

$\sqrt{s} = 2E$ - energija dostupna za stvaranje čestica

- stvorene čestice su p, p, π^0

$m_p = 938\ MeV$

$m_{\pi^0} = 135\ MeV$

$\Rightarrow \sqrt{s} = 2m_p + m_{\pi^0}$

u lab sustavu:

$$s = (p_1' + p_2')^2 = (E_1' + m_p)^2 - (\vec{p}_1' + 0)^2 = \underbrace{E_1'^2 + 2E_1' m_p + m_p^2 - |\vec{p}_1'|^2}_{m_p^2} = 2m_p(E_1' + m_p)$$

$$s = 4m_p^2 + 4m_p m_{\pi^0} + m_{\pi^0}^2 = 2m_p(E_1' + m_p)$$

$$\Rightarrow E_1' = \frac{2m_p^2 + 4m_p m_{\pi^0} + m_{\pi^0}^2}{2m_p} \approx 1218\ MeV$$

b) Treba pogledati za E_1' u SCM jer su čestice koje miruju

$\theta = 0$

$$\vec{p}_1 = \vec{p}_3 + \vec{p}_4 + \vec{p}_5 = \gamma(2m_p\vec{\beta} + m_{\pi^0}\vec{\beta}) \quad \text{zbog } \theta = 0 \text{ možemo zamijeniti } \vec{p}_1 = |\vec{p}_1| \ldots$$

koristili smo, $\vec{p} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} = m\vec{v}\gamma = m\gamma\beta$

$$\sqrt{E_1'^2 - m_p^2} = \gamma\beta(2m_p + m_{\pi^0}) \Rightarrow \gamma\beta = \frac{\sqrt{E_1'^2 - m_p^2}}{2m_p + m_{\pi^0}}$$

$$\Rightarrow |\vec{p}_5| = |\vec{p}_{\pi^0}| = m_{\pi^0}\gamma\beta = m_{\pi^0}\frac{\sqrt{E_1'^2 - m_p^2}}{2m_p + m_{\pi^0}} \approx 53\ MeV$$

2) Razmotrite raspad $\Delta^{++} \to p\pi^+$

a) Odredite orbitalnu kutnu količinu gibanja konačnog stanja, ako je paritet očuvan.

b) Odredite ovisnost vjerojatnosti raspada o kutu $\angle(\vec{J}_\Delta, \vec{p}_p)$ između spina Δ^{++} i impulsa protona $J_3(\Delta^{++}) = 3/2$

$J^P(\Delta^{++}) = \frac{3}{2}^+$

$J^P(p) = \frac{1}{2}^+$

$J^P(\pi^+) = 0^-$

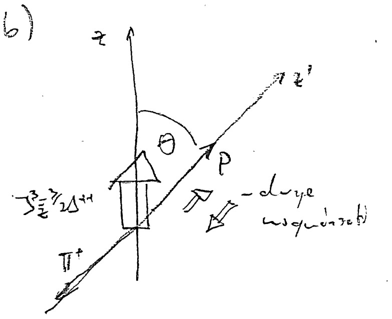
a) za očuvanje pariteta mora vrijediti

$$P(\Delta^{++}) = P(p)P(\pi^+)(-1)^L \Rightarrow (+1) = (+1)(-1)(-1)^L \Rightarrow L = \text{neparan}$$

- s druge strane $|j_1 - j_2| \leq j \leq |j_1 + j_2| \Rightarrow |\frac{1}{2} - L| \leq \frac{3}{2} \leq |\frac{1}{2} + L| \Rightarrow L = 1, 2$

slijedi da je $\boxed{L = 1}$

b)



- označimo vjerojatnost raspada s $W(\theta) = ?$
- prva stvar $\vec{L} = \vec{r} \times \vec{p}$ - prema tome nema angularnog momenta duž osi z'.
- doprinose samo spinovi p i π^+
- maksimalan spin od Δ^{++} stavljamo duž osi z

- raspisujemo spin od Δ^{++} po z' osi

$$|J=\tfrac{3}{2}, J_z=\tfrac{3}{2}\rangle = \sum_{J_{z'}} |J=\tfrac{3}{2}, J_{z'}\rangle \underbrace{\langle J=\tfrac{3}{2}, J_{z'} | J=\tfrac{3}{2}, J_z=\tfrac{3}{2}\rangle}_{d^{J=3/2}_{J_{z'}, J_z=3/2}(\theta)}$$

- konačna stanja su $_f\langle J_z=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2}|$ i $_f\langle J=\tfrac{3}{2}, J_{z'}=-\tfrac{1}{2}|$

1' mogući proces

$$_f\langle J_z=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2} | \underset{\text{čuva angularni moment}}{O} | J=\tfrac{3}{2}, J_z=\tfrac{3}{2}\rangle \overset{!}{=} \underbrace{_f\langle J_z=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2} | O | J_z=\tfrac{3}{2}, J_{z'}=\tfrac{1}{2}\rangle}_{A_{1/2}} d^{3/2}_{1/2\,3/2}(\theta)$$

vjerojatnost raspada je

$$W_1(\theta) = |A_{1/2}|^2 \left|d^{3/2}_{1/2\,3/2}(\theta)\right|^2$$

2' mogući proces

$$_f\langle J_z=\tfrac{3}{2}, J_{z'}=-\tfrac{1}{2} | O | J=\tfrac{3}{2}, J_z=\tfrac{3}{2}\rangle = \underbrace{\langle \tfrac{3}{2}, -\tfrac{1}{2} | O | \tfrac{3}{2}, -\tfrac{1}{2}\rangle}_{A_{-1/2}} d^{3/2}_{-1/2\,3/2}(\theta)$$

vjerojatnost raspada je

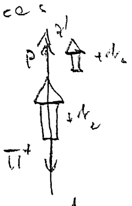
$$W_2(\theta) = |A_{-1/2}|^2 \left|d^{3/2}_{-1/2\,3/2}(\theta)\right|^2$$

- ukupna vjerojatnost raspada je

$$W(\theta) = W_1(\theta) + W_2(\theta) = \left[-\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}\right]^2 |A_{1/2}|^2 + \left[\sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}\right]^2 |A_{-1/2}|^2$$

- ako je paritet očuvan!

1' proces



$A_{1/2}$

2' proces

$A_{-1/2}$

pomoću operatora pariteta (zrcaljenjem)
za očuvani paritet moraju biti iste vjerojatnosti

$$|A_{1/2}|^2 = |A_{-1/2}|^2$$

3.) Razmotrite neutralne vektorske mezone ρ^0, ω, ϕ i J/ψ koji imaju $J^P = 1^-$.
Uz pretpostavku elektromagnetskog raspada na e^-e^+ par kroz kvark-antikvark anihilaciju preko virtualnog fotona, nađite omjere širina raspada:

$\rho^0 = \frac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})$ $\qquad Q(u) = Q(c) = +\frac{2}{3}$

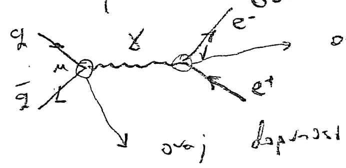
$\omega = \frac{1}{\sqrt{2}}(d\bar{d} + u\bar{u})$ $\qquad Q(d) = Q(s) = -\frac{1}{3}$

$\phi = s\bar{s}$

$J/\psi = c\bar{c}$

- traži se $\Gamma_{\rho^0} : \Gamma_\omega : \Gamma_\phi : \Gamma_{J/\psi} = ?$

- imamo procese oblika



ovaj vrh doprinosi $-ie\gamma^\mu$

ovaj doprinosi $-ieQ(q)\gamma^\mu$

- znači da imamo $M_{fi} \sim Q$
- odnosno $\Gamma \sim |M_{fi}|^2 \sim |Q|^2$
- za $\phi \to e^-e^+$ i $J/\psi \to e^-e^+$

$$\frac{\Gamma_\phi}{\Gamma_{J/\psi}} = \frac{|Q(s)|^2}{|Q(c)|^2} = \frac{|-\frac{1}{3}|^2}{|\frac{2}{3}|^2} = \frac{1}{4}$$

- što je s ρ^0 i ω

$\rho^0 \to e^+e^-$

$$M_{fi} = \langle f|S|i\rangle = \langle f|S|\tfrac{1}{\sqrt{2}}(d\bar{d} - u\bar{u})\rangle = \frac{1}{\sqrt{2}}\langle f|S|d\bar{d}\rangle - \frac{1}{\sqrt{2}}\langle f|S|u\bar{u}\rangle \sim \frac{1}{\sqrt{2}}Q(d) - \frac{1}{\sqrt{2}}Q(u)$$

$$\Gamma_{\rho^0} \sim \frac{1}{2}\left(\frac{1}{3}\right)^2 - \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^2$$

$\omega \to e^+e^-$

$$\Rightarrow \Gamma_\omega \sim \frac{1}{2}\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{2}{3}\right)^2$$

konačno dobijemo

$$\Gamma_\rho : \Gamma_\omega : \Gamma_\phi : \Gamma_{J/\psi} = 1 : 9 : 2 : 8$$

4.) Izračunajte:

$\sum_{s=1,2} v(p,s)\bar{v}(p,s)$ koristeći $v(p,s) = \sqrt{E+m}\begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E+m}\chi^s \\ \chi^s \end{pmatrix}$, $\chi^1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\chi^2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\sum_{s=1,2} v(p,s)\bar{v}(p,s) = (E+m)\left[\begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}\left((0\,1)\frac{\vec{\sigma}\vec{p}}{E+m} \quad (0\,1)\right) + \begin{pmatrix} \frac{\vec{\sigma}\vec{p}}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}\left((1\,0)\frac{\vec{\sigma}\vec{p}}{E+m} \quad (1\,0)\right)\right]\begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} =$$

$$= (E+m)\left[\begin{pmatrix} \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{\vec\sigma\vec p}{E+m} & \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}0&0\\0&-1\end{pmatrix} \\ \begin{pmatrix}0&0\\0&1\end{pmatrix}\frac{\vec\sigma\vec p}{E+m} & \begin{pmatrix}0&0\\0&-1\end{pmatrix}\end{pmatrix} + \begin{pmatrix} \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{\vec\sigma\vec p}{E+m} & \frac{\vec\sigma\vec p}{E+m}\begin{pmatrix}-1&0\\0&0\end{pmatrix} \\ \begin{pmatrix}1&0\\0&0\end{pmatrix}\frac{\vec\sigma\vec p}{E+m} & \begin{pmatrix}-1&0\\0&0\end{pmatrix}\end{pmatrix}\right] =$$

$$= (E+m)\begin{pmatrix} \frac{(\vec\sigma\vec p)^2}{(E+m)^2} & -\frac{\vec\sigma\vec p}{E+m} \\ \frac{\vec\sigma\vec p}{E+m} & -\mathbb{1}\end{pmatrix} = \begin{pmatrix} \frac{\vec p^{\,2}}{E+m} & -\vec\sigma\vec p \\ \vec\sigma\vec p & -(E+m)\mathbb{1}_{2\times2}\end{pmatrix} - m\mathbb{1}_{4\times4} + m\mathbb{1}_{4\times4} = \begin{pmatrix} E\mathbb{1}_{2\times2} & -\vec\sigma\vec p \\ \vec\sigma\vec p & -E\mathbb{1}_{2\times2}\end{pmatrix} - m\mathbb{1}_{4\times4} = \not{p} - m$$