Nuclear Theory I

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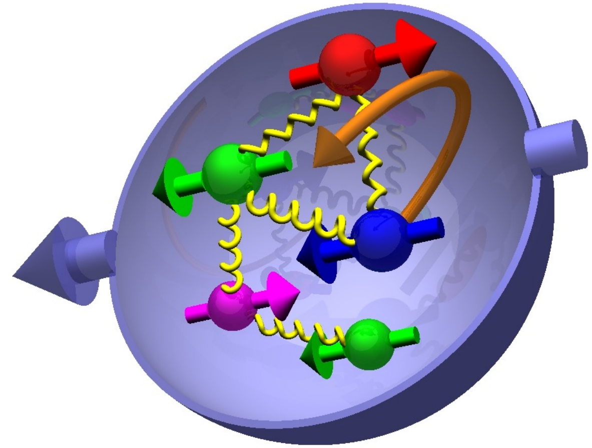
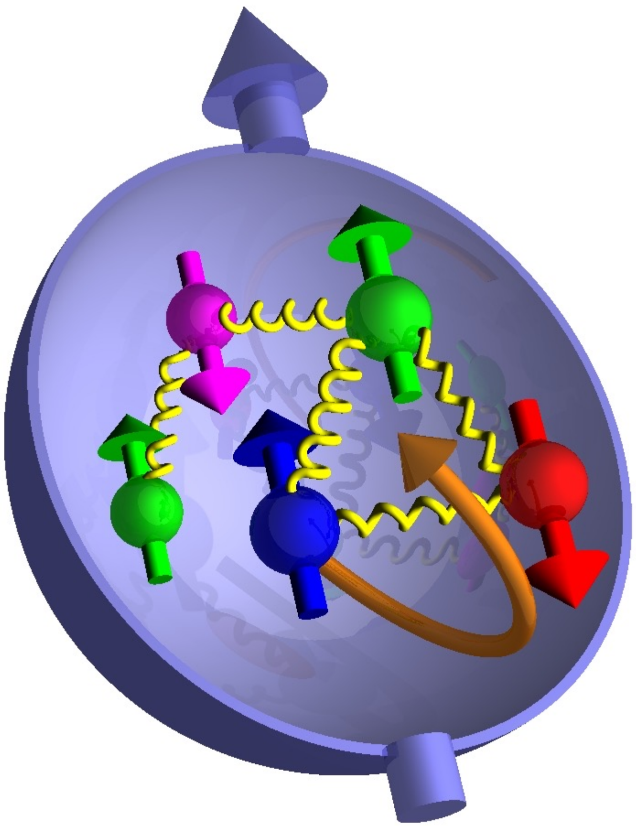




The Two-Nucleon System

The Deuteron

Data for the deuteron:

a) Binding energy: $E_B = 2.22464 \pm 0.00005$ MeV

(i) from the measurements of its atomic mass, and the comparison of the result with the sum of the masses of the proton and the neutron

(ii) measurement of the gamma ray energy emitted when the neutron and proton combine to form a bound state (n-p capture)

b) Angular momentum and parity: $J^\pi = 1^+$

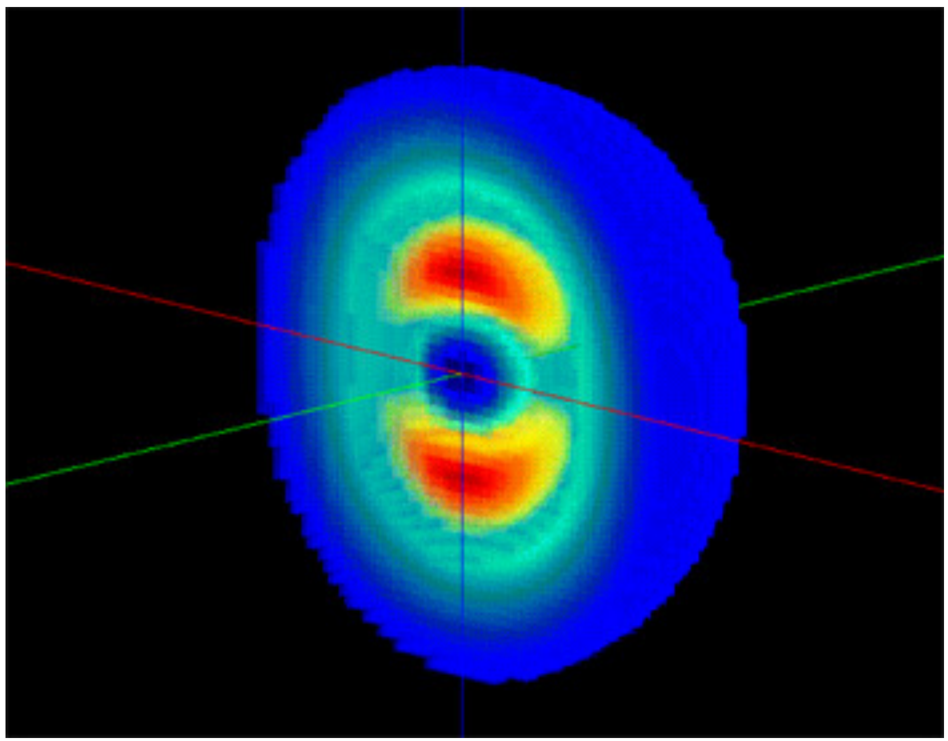
c) Magnetic dipole moment: $\mu_d = (0.857393 \pm 0.000001)\ \mu_N$

nuclear magneton $\mu_N = e\hbar/2m_N c$

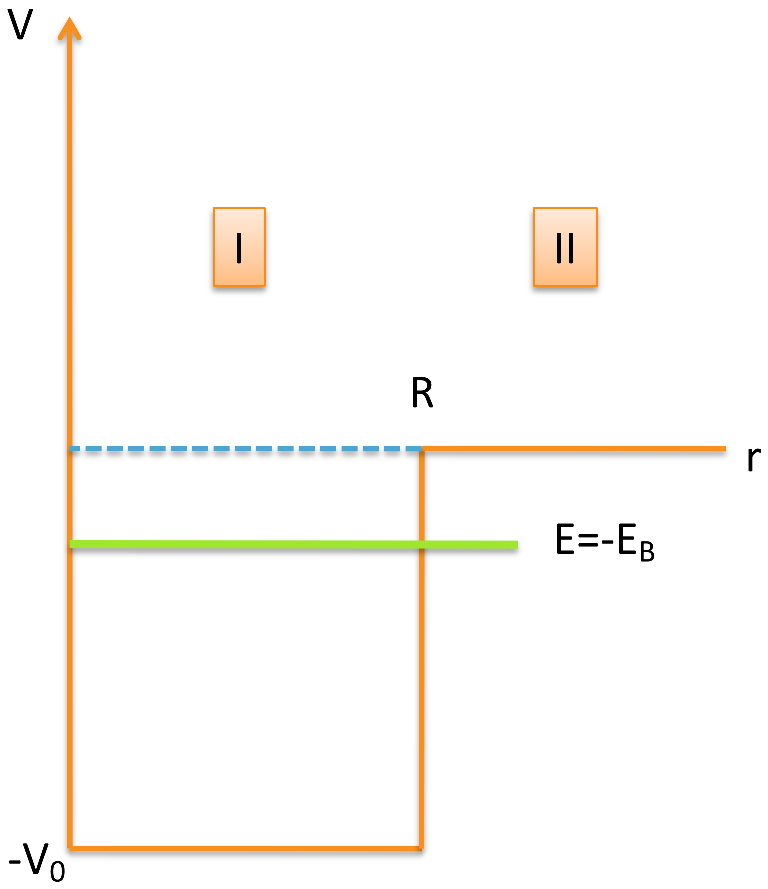
d) Electric quadrupole moment: $Q_d = 0.00282$ b (1b=10^{-28} m^2)

$$\frac{<z^2>}{<r^2>} \approx \frac{1.14}{3}$$

e) The radius of the deuteron: r_d=2.1 fm



A square-well model for the deuteron



In the center of mass system:

$$-\frac{\hbar^2}{2M}\nabla^2\psi + V(\vec{r})\psi = E\psi$$

-reduced mass:

$$\frac{1}{M} = \frac{1}{M_n} + \frac{1}{M_p}$$

For a spherically symmetric potential:

$$\psi = \sum_{lm} \frac{u_l(r)}{r} Y_l^m(\theta, \phi)$$

$$\frac{d^2 u_l}{dr^2} + \frac{2M}{\hbar^2}\left[E - V(r) - \frac{l(l+1)}{2Mr^2}\right] u_l = 0$$

REGION I

…for the l=0 state: $V(r) = -V_0$ $E=-E_B=-2.225$ MeV

$$\frac{d^2 u}{dr^2} + \frac{2M}{\hbar^2}\left[V_0 - E_B\right] u = 0$$

…boundary condition u=0 at r=0

$$\Rightarrow \quad u_I = A \sin(Kr) \qquad K = \frac{1}{\hbar}\sqrt{2M(V_0 - E_B)}$$

REGION II

$$\frac{d^2u}{dr^2} - \frac{2M}{\hbar^2}E_B u = 0$$

…boundary condition u=0 at r=∞

$$u_{II} = Be^{-kr} \qquad k = \frac{1}{\hbar}\sqrt{2ME_B}$$

… u and du/dr continuous at r=R:

$$A\sin(KR) = Be^{-kR} \qquad AK\cos(KR) = -kBe^{-kR}$$

$$K\cot(KR) = -k$$

…implicit relation between E_B, the width R and depth V_0 of the potential.

EXP. E_B=-2.225 MeV. For R=2.1 fm (exp. charge radius), the numerical solution:

$$V_0 \approx 34\ \mathrm{MeV}$$

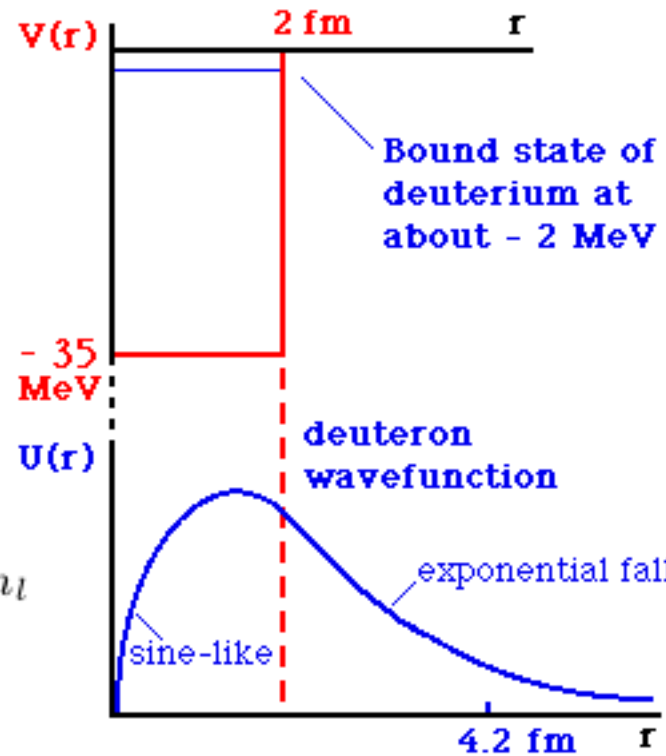
For this value of V_0 the equation has only one solution for E_B. There are no excited bound states.

The deuteron wave function

$Q_d \neq 0$

$\mu_d \neq \mu_n + \mu_p$

The deuteron ground state cannot be described by the spherically symmetric $^3S_1$ wave function.



$$\mathcal{Y}_{lSJ}^{M} = \sum_{m_s m_l} < lSm_l m_s | JM > \chi_S^{m_s} Y_l^{m_l}$$

… states with maximal projection:

$$^3S_1 : \qquad \psi_S \equiv \psi_{011}^1 = \frac{u_0(r)}{r}\mathcal{Y}_{011}^1 = \frac{u_0(r)}{r} Y_0^0 \chi_1^1$$

$$^3D_1 : \qquad \psi_D \equiv \psi_{211}^1 = \frac{u_2(r)}{r}\mathcal{Y}_{211}^1 =$$

$$\frac{u_2(r)}{r}\left[\sqrt{\frac{3}{5}} Y_2^2 \chi_1^{-1} - \sqrt{\frac{3}{10}} Y_2^1 \chi_1^0 + \sqrt{\frac{1}{10}} Y_2^0 \chi_1^1\right]$$

The magnetic moment operator:

$$\vec{\mu} = \sum_a \left(g_a^s \vec{s}_a + g_a^l \vec{l}_a \right)$$

$$g_p^l = 1 \quad g_n^l = 0 \quad g_p^s = 5.58 \quad g_n^s = -3.82$$

In the c.m.s. l_p=1/2 l (relative orbital angular momentum). The magnetic moment of the deuteron is defined as:

$$\mu = < \psi_{lSJ}^{M=J} | \mu_z | \psi_{lSJ}^{M=J} >$$

$${}^3S_1 : \quad < \psi_{011}^1 | \mu_z | \psi_{011}^1 >= \frac{g_p^s + g_n^s}{2} \int_0^\infty u_0^2(r) dr =$$

$$\frac{g_p^s + g_n^s}{2} = 0.88 \mu_N$$

$${}^3D_1 : \quad < \psi_{211}^1 | \mu_z | \psi_{211}^1 >= \frac{3}{4} - \frac{g_p^s + g_n^s}{4} = 0.31 \mu_N$$

$$\text{EXP.} \quad \mu_d = 0.8573 \mu_N$$

The ground-state wave function is a superposition of the $^3S_1$ and $^3D_1$ states.

$$\psi = C_S \psi_S + C_D \psi_D \qquad C_S^2 + C_D^2 = 1$$

$$\underline{\mu_d = C_S^2\ 0.88\mu_N + C_D^2\ 0.31\mu_N = 0.8573\mu_N}$$

$$C_S^2 = 0.96 \qquad C_D^2 = 0.04$$

$$\psi_d = C_S \frac{u_0}{r} \mathcal{Y}_{011} + C_D \frac{u_2}{r} \mathcal{Y}_{211}$$

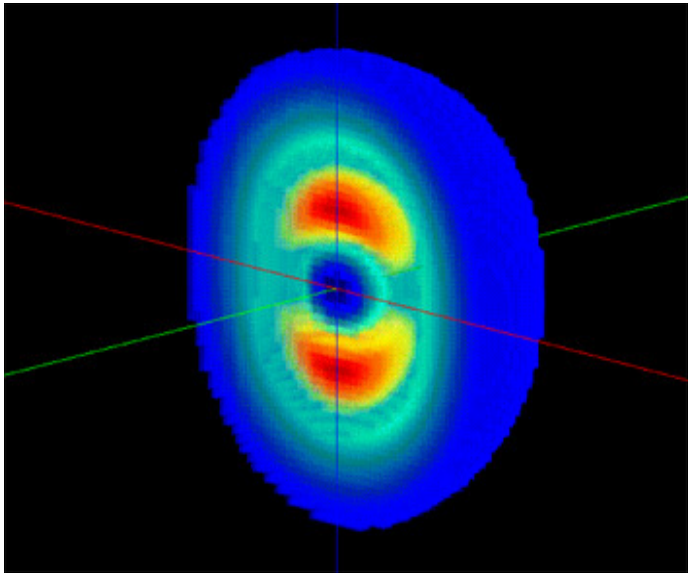
The electric quadrupole moment:

… the quadrupole operator:

$$Q_0 = e(3z^2 - r^2) = \sqrt{\frac{16\pi}{5}} e r^2 Y_{20}(\theta, \phi)$$

The quadrupole moment is defined as: $< \psi_d; JM = J|Q_0|\psi_d; JM = J >$

Q_0 is an operator in coordinate space and does not depend on the spin. It is a tensor operator of rank 2 and $<Q_0> \neq 0$ only for $l \geq 1$. This means that $<Q_0> = 0$ in the $^3S_1$ state. The quadrupole moment of the deuteron presents direct evidence for the presence of the $^3D_1$ component in the ground-state wave function.



EXP. $Q_d = 0.28$ e fm^2

… positive value -> prolate shape. Elongated along the z-axis (axial symmetry).

The calculated value:

$$Q_d(^3D_1) = -\frac{1}{5}e < r^2 >_D \approx -0.77 \text{ e fm}^2$$

for the exp. value of the charge radius. <u>Even the sign is wrong!</u>

$$Q_d = C_S^2 < {}^3S_1|Q_0|^3S_1 > + C_D^2 < {}^3D_1|Q_0|^3D_1 > + 2C_SC_D < {}^3S_1|Q_0|^3D_1 >$$

The first term vanishes, the second is small (~ C_D^2) and with a wrong sign. The third, non-diagonal term dominates.

Low-energy scattering

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta)$$

… in the limit of very low energy, contribution only from the l=0 term ($P_0(x)=1$):

$$f(\theta) = \frac{1}{k}e^{i\delta_0}\sin\delta_0 \quad \Rightarrow \quad \sigma = \frac{4\pi}{k} Im f(0) = \frac{4\pi}{k^2}\sin^2\delta_0$$

… in the extreme limit E -> 0, f(θ) remains finite only if δ_0 -> 0

$$\boxed{\lim_{E\to 0} f(\theta) = \lim_{k\to 0}\frac{\delta_0}{k} = -a}$$

scattering length

$\Rightarrow$ $\sigma = 4\pi a^2$ and $u_0 \sim \sin(kr+\delta_0) \approx kr + \delta_0 = \underline{k(r-a)}$

can be used to determine if a state is bound!

E. Wigner -> the nuclear force depends on the spin. NN scattering differs when nucleons collide with parallel (triplet) spins or antiparallel (singlet) spins.

… NN scattering without polarization:

$$\sigma = \frac{3}{4}\sigma_t + \frac{1}{4}\sigma_s$$

Neutron-proton scattering cross section:

If σ = 20.4 b and σ_t = 3.4 b, then σ_s = 71 b.

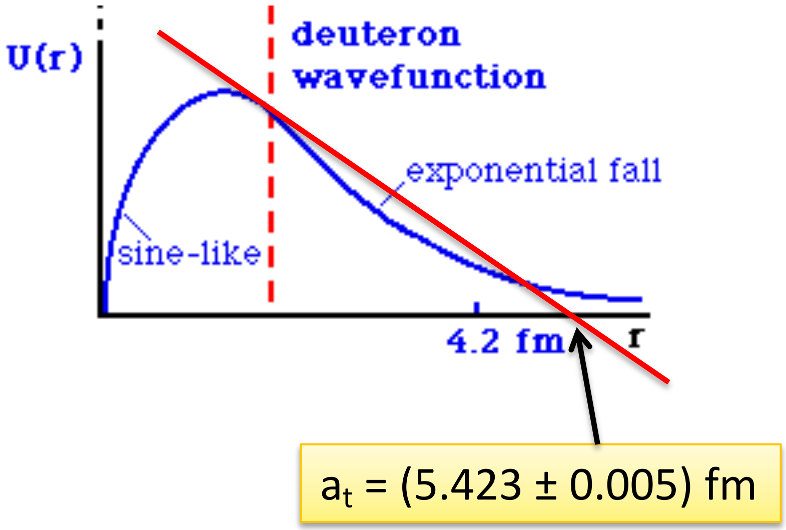
$$\sigma = \frac{3}{4} 4\pi a_t^2 + \frac{1}{4} 4\pi a_s^2$$

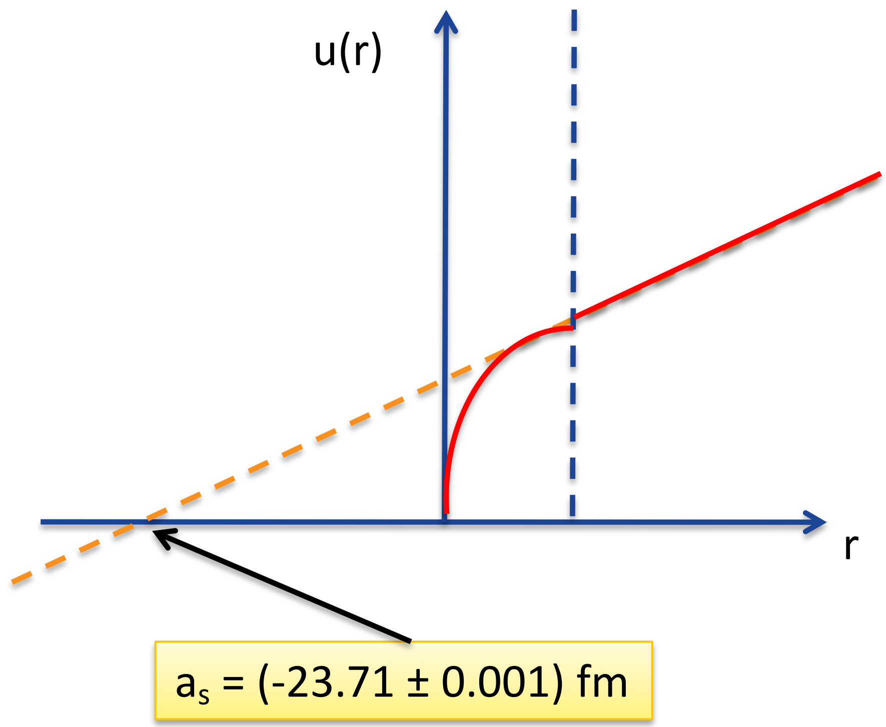
This is because the singlet potential is shallower that the triplet one, and close to the threshold for the appearance of the first bound state. This leads to a resonance when the incident particle has very low energy.

a_t = (5.423 ± 0.005) fm a_s = (-23.71 ± 0.001) fm

a > 0 existence of a bound state, a < 0 no bound state!

… in the asymptotic region: $u_0(r) \approx k(r-a)$





Effective Range Theory

What happens when the E -> 0 approximation is no longer valid?

$l = 0$ in n-p scattering presents a safe approximation up to ~ 20 MeV.

… consider an incident neutron with energy E_1 and wave number: $k_1 = \frac{\sqrt{2mE_1}}{\hbar}$

… radial equation for l=0:

$$\frac{d^2u_1}{dr^2} + k_1^2 u_1 - \frac{2m}{\hbar^2} V(r) u_1 = 0$$

… for another energy E_2:

$$\frac{d^2u_2}{dr^2} + k_2^2 u_2 - \frac{2m}{\hbar^2} V(r) u_2 = 0$$

$$u_2 \frac{d^2u_1}{dr^2} - u_1 \frac{d^2u_2}{dr^2} = (k_2^2 - k_1^2) u_1 u_2$$

$$u_2 u_1' - u_1 u_2' \big|_0^R = (k_2^2 - k_1^2) \int_0^R u_1 u_2 \, dr \qquad \star$$

… in the asymptotic region:

$$u_l(r) \sim \sin(kr - l\pi/2 + \delta_l)$$

Let ψ be the asymptotic form of u for l=0, but valid for every point in space:

$$\psi_1(r) = \frac{\sin(k_1 r + \delta_1)}{\sin \delta_1} \qquad \psi_2(r) = \frac{\sin(k_2 r + \delta_2)}{\sin \delta_2}$$

normalization

$$\psi_2 \psi_1' - \psi_1 \psi_2' \big|_0^R = (k_2^2 - k_1^2) \int_0^R \psi_1 \psi_2 \, dr \qquad \star\star$$

R is arbitrary. If it is chosen beyond the range of the nuclear potential ψ(R) = u(R), and the LHS of ★ and ★★ coincide at r=R. In addition $u_1(0)=u_2(0)=0$. For R -> ∞ and ★★ – ★:

$$\psi_1(0)\psi_2'(0) - \psi_2(0)\psi_1'(0) = (k_2^2 - k_1^2)\int_0^\infty (\psi_1\psi_2 - u_1u_2)\, dr$$

$$\Rightarrow \quad k_2 \cot\delta_2 - k_1 \cot\delta_1 = (k_2^2 - k_1^2)\int_0^\infty (\psi_1\psi_2 - u_1u_2)\, dr$$

Consider the special case $k_1 \to 0$: $$k_1 \cot\delta_1 = k_1 \frac{\cos\delta_1}{sin\delta_1} \approx -\frac{1}{a}$$

$$\Rightarrow \quad k \cot\delta = -\frac{1}{a} + k^2 \underbrace{\int_0^\infty (\psi_0\psi - u_0u)\, dr}_{\neq 0 \text{ only inside the range of the potential}}$$

… approximation $\psi \approx \psi_0$ and $u \approx u_0$, because in both cases the energy is small compared to V.

$$\Rightarrow \quad k \cot\delta = -\frac{1}{a} + \frac{1}{2}k^2 r_{eff}$$

$$r_{eff} = 2\int_0^\infty (\psi_0^2 - u_0^2)\, dr$$

EFFECTIVE RANGE

… cross section:

$$\sigma = \frac{4\pi}{k^2}\sin^2\delta_0 = 4\pi\frac{1}{k^2 + k^2\cot^2\delta_0}$$

$$\sigma = \frac{4\pi a^2}{a^2k^2 + (1 - \frac{1}{2}a\, r_{eff}k^2)^2}$$

The effect of the potential is parameterized with the effective range r_{eff} and the scattering length a.

Low-energy scattering does not provide information about the form of the NN potential.

n-p scattering:

$$\sigma = \frac{3}{4}\frac{4\pi a_t^2}{a_t^2k^2 + (1 - \frac{1}{2}a_t r_t k^2)^2} + \frac{1}{4}\frac{4\pi a_s^2}{a_s^2k^2 + (1 - \frac{1}{2}a_s r_s k^2)^2}$$

4 parameters: a_t, a_s, r_t, r_s

a_t = (5.423 ± 0.005) fm a_s = (-23.71 ± 0.001) fm

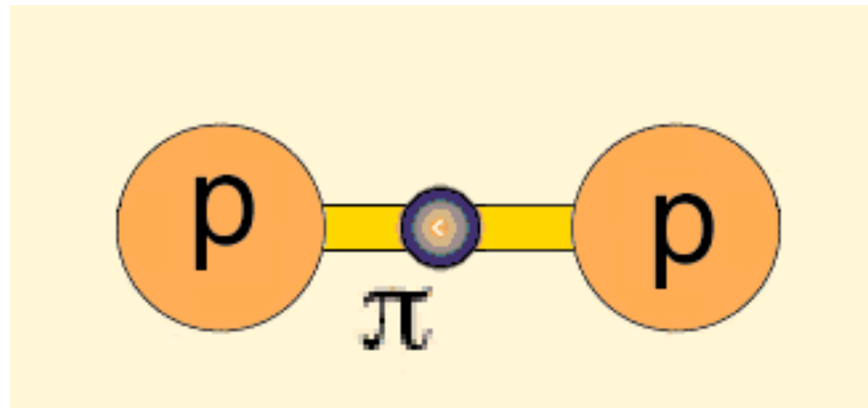
r_{eff} in the triplet state can be obtained from the binding energy of the deuteron:

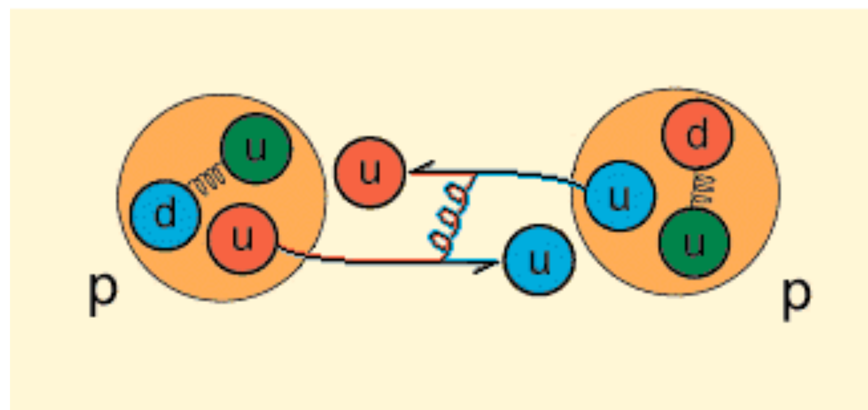
r_t = 1.76 fm

From the fit of the cross section to low-energy scattering data:

r_s = 2.56 fm

The NN Interaction





Nucleons are basic components of nuclei. A traditional goal of nuclear physics has been the understanding of the properties of atomic nuclei in terms of the “bare” interaction between pairs of nucleons. However, the underlying theory of strong interactions, QCD, shows that the NN interaction is not fundamental.

… nucleon degrees of freedom: $\vec{r}_i, \quad \vec{p}_i, \quad \vec{s}_i, \quad \vec{t}_i \quad (i = 1, \dots A)$

Charge independence of the NN interaction

$$V_{np} = V_{pp} = V_{nn}$$

→ isospin quantum number

The wave function of the NN system: $\Psi_{NN} = \psi_{\text{space}} \chi_S^{m_s} \phi_T^{T_z}$

Phenomenological potentials

An appropriate functional form for the NN potential is parameterized in such a way that it reproduces as closely as possible the data on NN scattering and deuteron properties. There are two classes of such potentials: <u>local and nonlocal.</u>

A local potential is completely specified at each point **r** in space. An example of nonlocal potentials are momentum-dependent potentials.

LOCAL POTENTIALS

$$V(1,2) \equiv V(\vec{r}_j, \vec{p}_j, \vec{\sigma}_j, \vec{\tau}_j; j = 1, 2)$$

Symmetry and invariance properties of the Hamiltonian operator constrain the general form of the interaction.

… if we consider the proton and neutron masses to be equal, the cms and relative coordinates:

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{R} = \frac{1}{2}(\vec{r}_1 + \vec{r}_2)$$

$$\vec{p} = \frac{1}{2}(\vec{p}_1 - \vec{p}_2) \qquad \vec{P} = \vec{p}_1 + \vec{p}_2$$

1) invariance under translations $\qquad \vec{r}_j \to \vec{r}_j + \vec{a} \quad j = 1, 2$

$$\Rightarrow \quad V(\vec{r}, \vec{p}, \vec{R}, \vec{P}) = V(\vec{r}, \vec{p}, \vec{R} + \vec{a}, \vec{P})$$

2) invariance under Galilean transformations $\vec{p}_j \rightarrow \vec{p}_j + \vec{p}_0 \quad j = 1, 2$

$$\Rightarrow \quad V(\vec{r}, \vec{p}, \vec{R}, \vec{P}) = V(\vec{r}, \vec{p}, \vec{R}, \vec{P} + 2\vec{p}_0)$$

$\vec{a}, \vec{p}_0$ can take any value $\Rightarrow$ $V(1,2) \neq V(\vec{R}, \vec{P})$

$$V(1,2) = V(\vec{r}, \vec{p}, \vec{\sigma}_j, \vec{\tau}_j; j = 1, 2)$$

3) Space reflection and time-reversal invariance. Under space inversion (**r** -> -**r**) and time reversal (**σ** -> -**σ**), **σ** and **r** are not invariant separately. They may appear in the potential only in product forms. The parity of a closed system is conserved in strong and electro-magnetic interactions.

4) Rotational invariance

5) Particle exchange symmetry (Pauli principle) $V(1,2) = V(2,1)$

The most general form of the NN potential that preserves invariance under particle exchange, translation, Galilean transformation, rotation, parity, and time-reversal:

$$V(1,2) = V_C + V_S(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + V_T S_{12}(\vec{r}) + V_{T'} S_{12}(\vec{p}) + V_{LS} \vec{L} \cdot \vec{S} + V_Q (\vec{L} \cdot \vec{S})^2$$

… the spin-orbit operator: $\vec{L} \cdot \vec{S} = \frac{1}{2}(\vec{r} \times \vec{p}) \cdot (\vec{\sigma}_1 + \vec{\sigma}_2)$

… the tensor operator $S_{12} = 3\left(\vec{\sigma}_1 \cdot \frac{\vec{r}}{r}\right)\left(\vec{\sigma}_2 \cdot \frac{\vec{r}}{r}\right) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)$

… scalar functions: $V_\alpha = V_\alpha(r^2, p^2, L^2) \qquad \alpha = \{C, S, T, T', LS, Q\}$

Isospin dependence of the interaction: $[H, T_\pm] = 0 \qquad [H, T_z] = 0$

⇒ $[H, T^2] = 0$ States with T=0 and T=1 correspond to different energies, even though nuclear forces do not depend on the charge (isospin). This is because of the dependence of the nuclear force <u>on the spin</u> (the total wave function must be antisymmetric).

V_α must be scalars in isospin space: $V_\alpha = V_{\alpha 0} + V_{\alpha 1} \vec{\tau}_1 \cdot \vec{\tau}_2$

NONLOCAL POTENTIALS

The most general form of a potential: $-\frac{\hbar^2}{2\mu} p^2 |\psi> + \hat{V}|\psi> = E|\psi>$

$$\mu = \frac{M_p M_n}{M_p + M_n} \equiv \frac{1}{2} M \quad \Rightarrow \quad -\frac{\hbar^2}{M}\nabla^2\psi(\vec{r}) + <\vec{r}\,|\hat{V}|\psi> = E\psi(\vec{r})$$

$$<\vec{r}\,|\hat{V}|\psi> = \int d^3r' \;<\vec{r}\,|\hat{V}|\vec{r}\,'><\vec{r}\,'\,|\psi> = \int d^3r' V(\vec{r},\vec{r}\,')\psi(\vec{r}\,')$$

Local potentials present a special case: $V(\vec{r},\vec{r}\,') = V(\vec{r})\delta(\vec{r}-\vec{r}\,')$

$\Rightarrow$ $<\vec{r}\,|\hat{V}|\psi> = V(\vec{r}\,)\psi(\vec{r}\,)$

The action of the interaction at **r** depends only on the value of ψ at that point.

Meson-exchange potentials

In 1935 Yukawa made an analogy between the strong, short-ranged nuclear force and the electromagnetic force between charged particles. If the Coulomb force is due to the exchange of a virtual quantum – photon, perhaps the nuclear force is likewise due to a virtual particle, necessarily of integral spin, exchanged between nucleons.

The stationary Klein-Gordon equation for the pion field:

$$\left(\nabla^2 - \frac{m_\pi^2 c^2}{\hbar^2}\right)\phi = -g\delta(\vec{r})$$

solution: $$\phi = g\frac{e^{-\mu r}}{r} \qquad \mu \equiv \frac{m_\pi c}{\hbar}$$

The potential between two nucleons is proportional to the wave function of the pion, i.e. to the probability amplitude that the emitted pion finds itself close to the other nucleon.

YUKAWA POTENTIAL: $$\boxed{V = g^2\frac{e^{-\mu r}}{r}}$$

The one-pion exchange potential (OPEP)

π – pseudoscalar particle. The interaction Hamiltonian for two nucleons that interact by exchanging a pion:

$$H' = \frac{f}{\mu}\sum_{i=1}^{2}\int d^3r\ \delta(\mathbf{r}-\mathbf{r_i})\vec{\tau}_i\cdot(\sigma_i\cdot\nabla\vec{\phi}(\mathbf{r}))$$

$$\mu \equiv \frac{m_\pi c}{\hbar} \qquad \frac{f^2}{4\pi\hbar c} = 0.088 \pm 0.001$$

$\vec{\phi} \equiv \{\pi^+, \pi^-, \pi^0\}$ vector in isospin space

Klein-Gordon eq. $$(\nabla^2-\mu^2)\vec{\phi}(\mathbf{r}) = \frac{f}{\mu}\sum_{i=1}^{2}\vec{\tau}_i(\sigma_i\cdot\nabla_i)\delta(\mathbf{r}-\mathbf{r}_i)$$

… the pion field produced by nucleon 2 (source):

From the relation: $$(\nabla^2-\mu^2)\frac{e^{-\mu|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}')$$

$$\vec{\phi}(\mathbf{r}) = -\frac{f}{4\pi\mu}\vec{\tau}_2(\sigma_2 \cdot \nabla_2)\frac{e^{-\mu|\mathbf{r}-\mathbf{r}_2|}}{|\mathbf{r}-\mathbf{r}_2|}$$

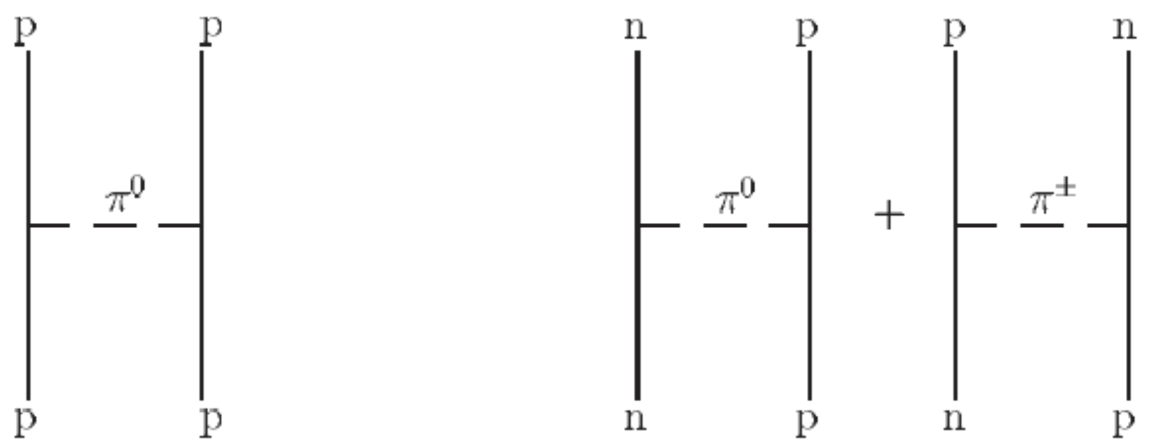
The interaction energy of nucleon 1 with this field generated by nucleon 2:

$$H' = -\frac{f^2}{4\pi\mu^2}(\vec{\tau}_1 \cdot \vec{\tau}_2)(\sigma_1 \cdot \nabla_1)(\sigma_2 \cdot \nabla_2)\frac{e^{-\mu|\mathbf{r}_1-\mathbf{r}_2|}}{|\mathbf{r}_1-\mathbf{r}_2|}$$

This is to be identified with the one-pion exchange potential:

$$V_{OPEP} = \frac{1}{3}m_\pi c^2\left(\frac{f^2}{4\pi\hbar c}\right)(\vec{\tau}_1 \cdot \vec{\tau}_2)\{(\sigma_1 \cdot \sigma_2) + S_{12}[1 + 3(\mu r)^{-1} + 3(\mu r)^{-2}]\}\frac{e^{-\mu r}}{\mu r}$$

The exchange of a pseudoscalar meson leads to a spin-dependent potential and a tensor part.



The OPEP potential describes the NN scattering for angular momenta l ≥ 6. This shows that OPEP represents the nuclear force at large distances (r ≥ 2 fm).

Generalized one-boson exchange:

… system of coupled nucleon and meson fields: $H = H_N^0 + H_m^0 + H_{mN}$

The meson-nucleon couplings:

1) scalar meson: $H_{mN}^{(S)} = g_s \bar{\psi}\psi\phi^{(S)}$

2) pseudoscalar meson: $H_{mN}^{(PS)} = i g_{ps} \bar{\psi}\gamma_5\psi\phi^{(PS)}$ pseudoscalar coupling

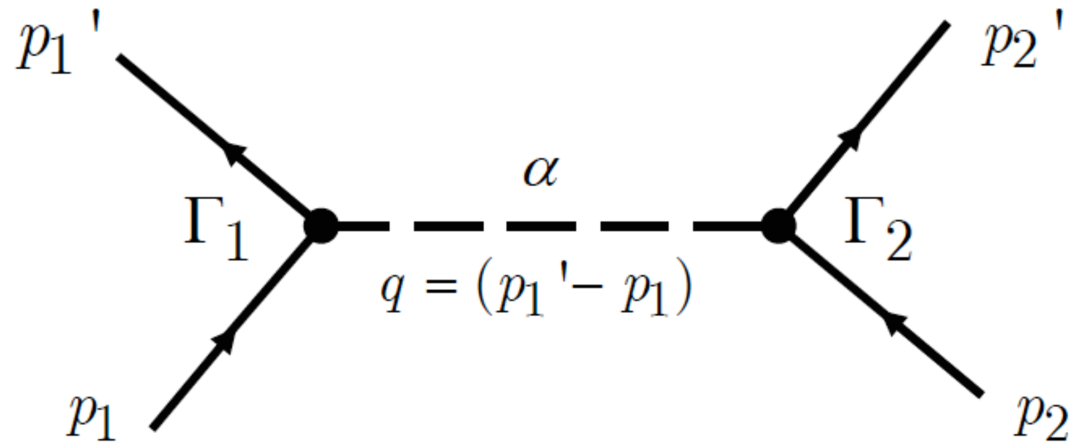
$$H_{mN}^{(PV)} = i \frac{f_{ps}}{m_{ps}} \bar{\psi}\gamma_5\gamma^\mu\psi\partial_\mu\phi^{(PS)}$$

pseudovector coupling

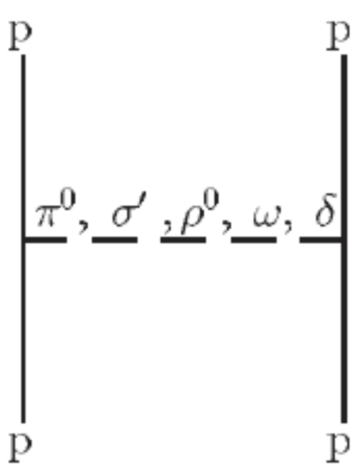
3) vector meson:

$$H_{mN}^{(V)} = g_v \bar{\psi}\gamma^\mu\psi\phi_\mu^{(V)} - \frac{f_v}{4M}\bar{\psi}\sigma^{\mu\nu}\psi(\partial_\mu\phi_\nu^{(V)} - \partial_\nu\phi_\mu^{(V)})$$

Feynman diagram for one-boson exchange:



Bonn potential: based on exchange of mesons for the NN interaction below the pion-production threshold.



In each spin-isospin channel the potential is written in the form:

$$V = V_C + V_T S_{12} + V_{LS} \vec{L} \cdot \vec{S}$$

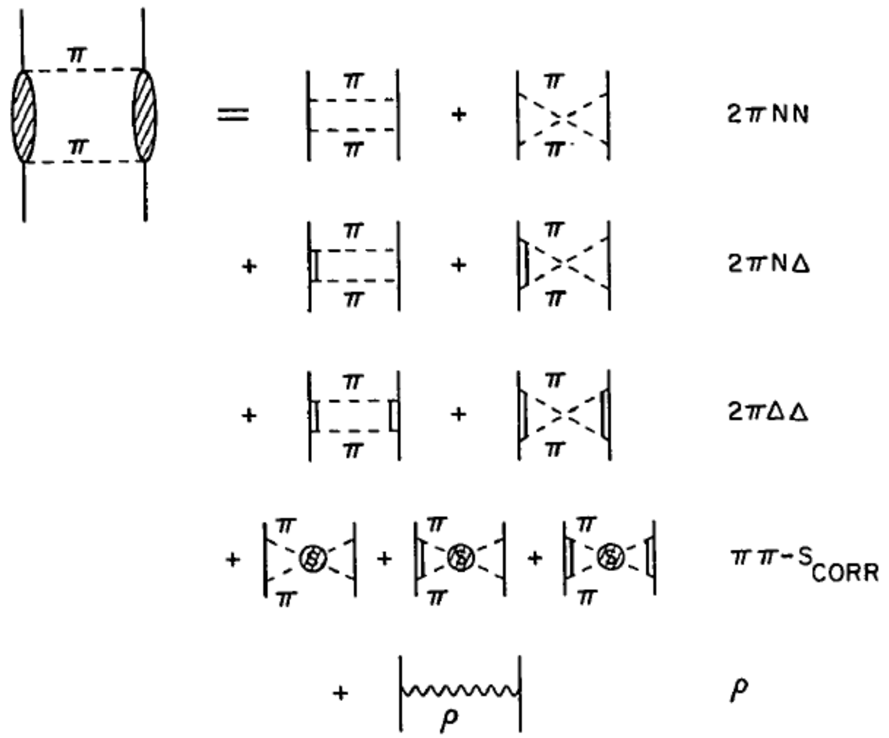
Vertex functions (form factors) are introduced to account for the finite size of the nucleons:

$$F(q^2) = \left(\frac{\Lambda^2 - m^2}{\Lambda^2 - q^2} \right)^{n/2} \qquad n = 1, 2, \ldots$$

⇒ the high-momentum components are suppressed.

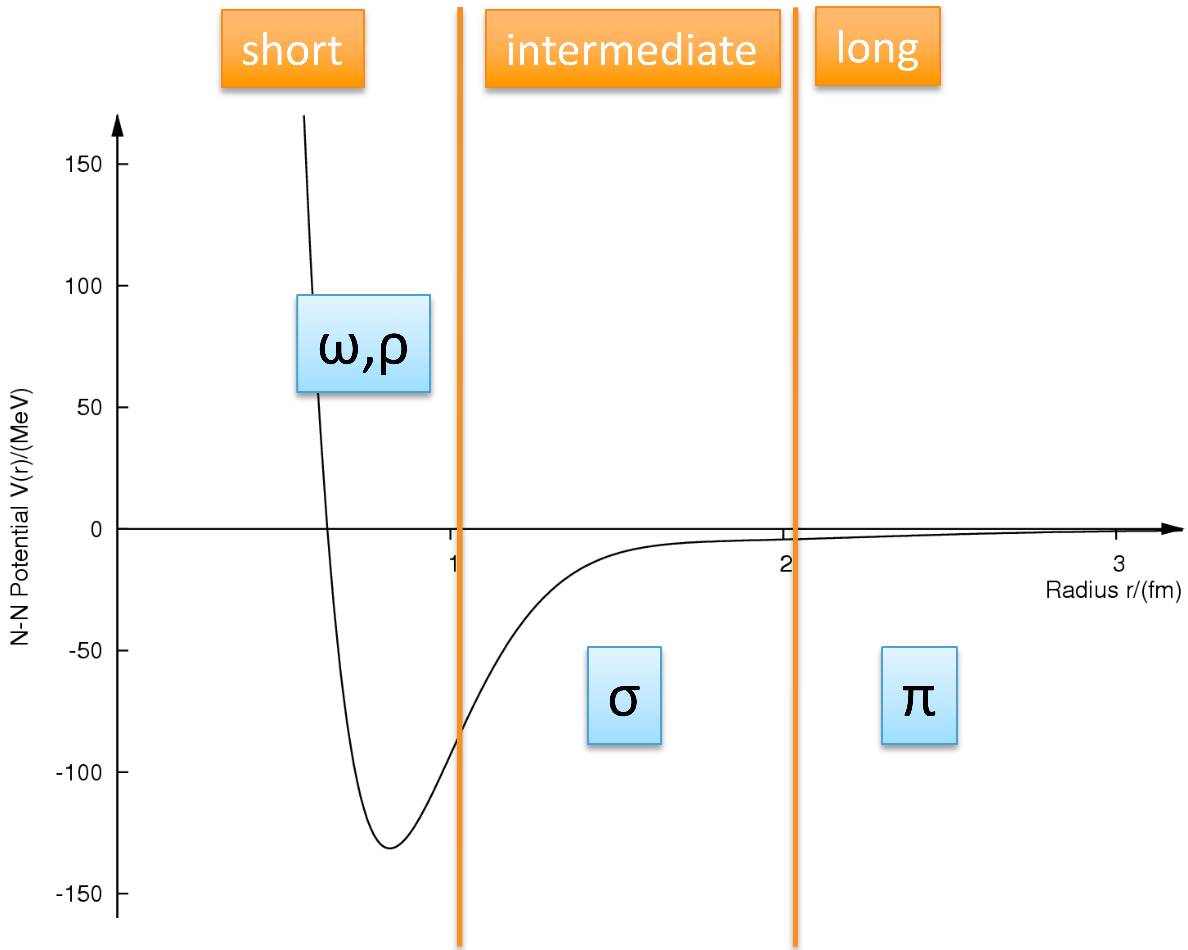
… the long range part of the potential is described by one-pion exchange (r > 2 fm).

… the intermediate-range part is attractive: two-pion exchange (TPE)



… the short-range part is dominated by vector meson exchange (ρ and ω) - repulsion

… the very short-range part of the potential is described purely phenomenologically, either by a sharp cut-off radius (*hard core*), or in a soft form (*soft core*)



short
intermediate
long
150
100
50
0
-50
-100
-150
N-N Potential V(r)/(MeV)
1
2
3
Radius r/(fm)
ω,ρ
σ
π

PWA

Models based on a potential do not use experimental data directly. Instead, they compare their outputs with phase shifts from partial wave analysis (PWA).

χ^2/datum

	CD-Bonn potential	Nijmegen phase shift analysis	Argonne V_{18} potential
	proton-proton data		
1992 *pp* database (1787 data)	1.00	1.00	1.10
After-1992 *pp* data (1145 data)	1.03	1.24	1.74
1999 *pp* database (2932 data)	1.01	1.09	1.35

Three- and Four-Nucleon Systems

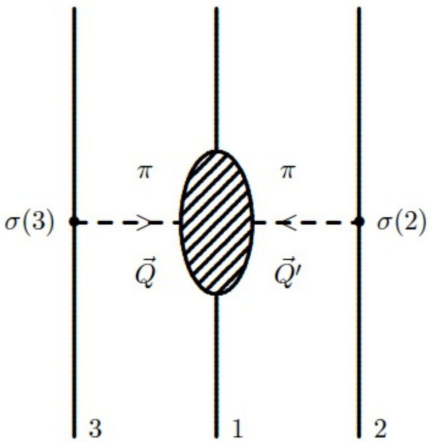
Potentials	$E(^3\text{H})$	$E(^3\text{He})$	$E(^4\text{He})$
Nijm 93	-7.668	-7.014	-24.53
Nijm I	-7.741	-7.083	-24.98
Nijm II	-7.659	-7.008	-24.56
AV18	-7.628	-6.917	-24.28
CD-Bonn	-8.013	-7.288	-26.26
Exp.	-8.482	-7.718	-28.30

^{3}H, ^{3}He and ^{4}He binding energy predictions for several NN potential models compared to the experimental values. All energies are given in MeV.

… underbinding – evidence for three-nucleon (3N) forces

The number of possible operators that can be used to construct a 3NF is much larger than in the NN-force case, and one cannot examine all of them to determine which are the important ones at low-energy.

2π – exchange 3NF



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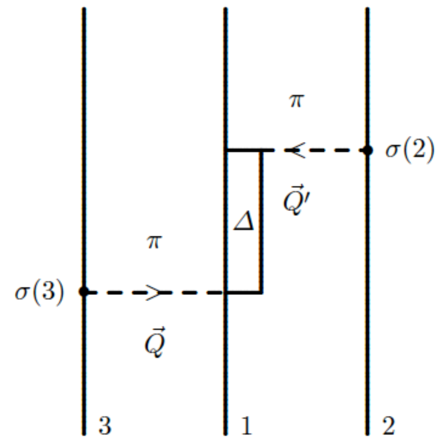
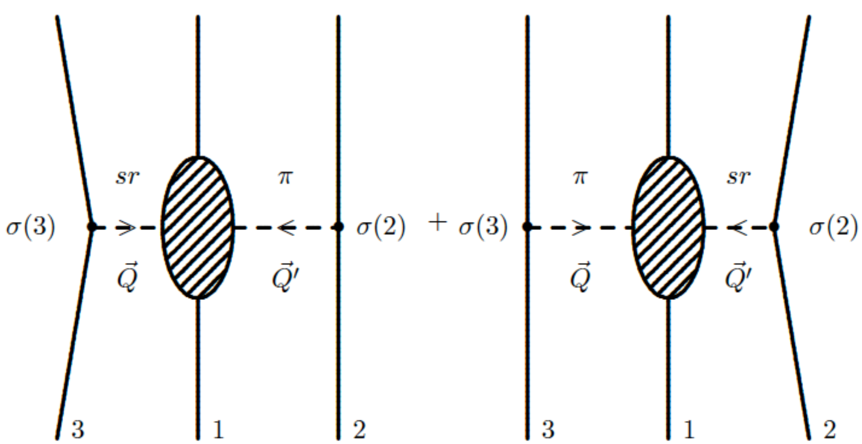


+ ...

The Fujita-Miyazawa 3NF.

3NF including heavier mesons

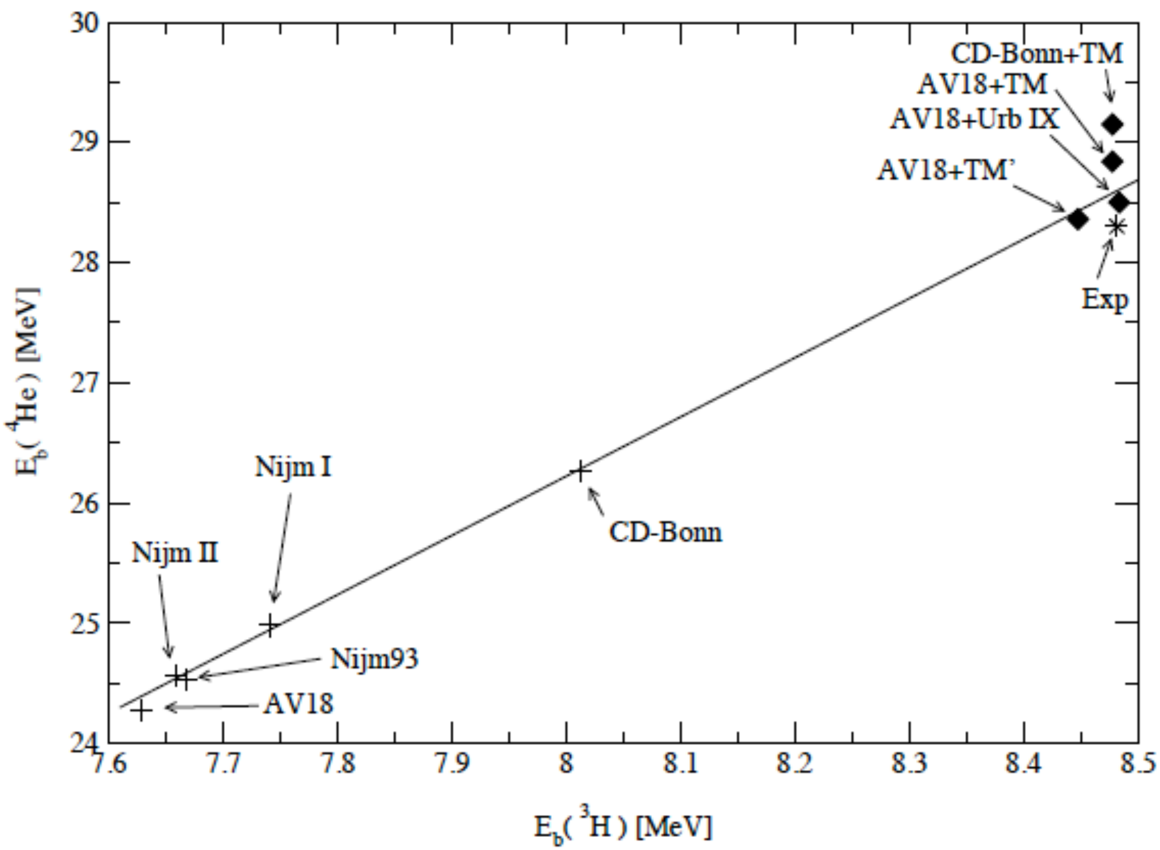


π-range – short-range 3NF terms.

Potentials	$\Lambda\ [m_\pi]$	$E(^3\mathrm{H})$	$E(^3\mathrm{He})$	$E(^4\mathrm{He})$
CD-Bonn+TM	4.784	-8.478	-7.735	-29.15
AV18+TM	5.156	-8.478	-7.733	-28.84
AV18+TM'	4.756	-8.448	-7.706	-28.36
AV18+Urbana IX	—	-8.484	-7.739	-28.50
Exp.	—	-8.482	-7.718	-28.30

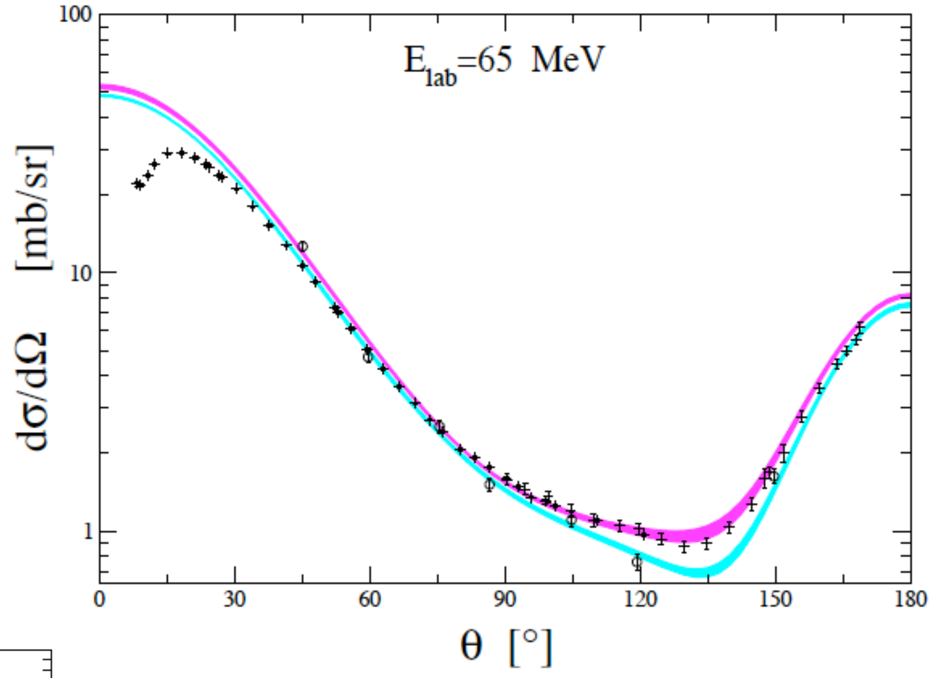
^{3}H, ^{3}He and ^{4}He binding energy predictions for several NN and 3N potential models compared to the experimental values. All energies are given in MeV.

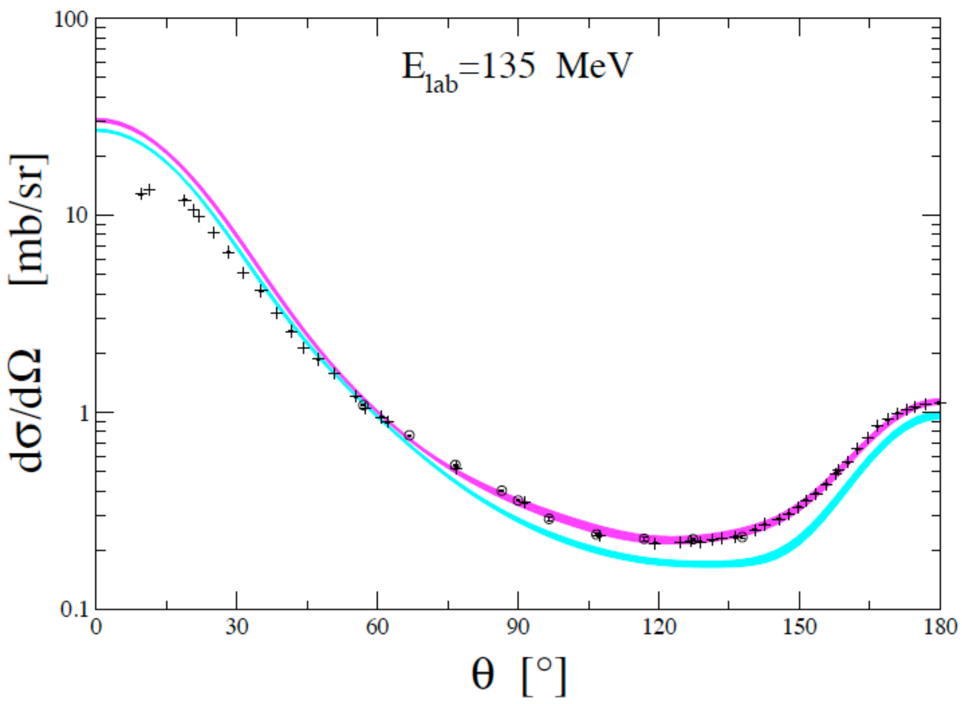
Tjon-line: α-particle binding energy predictions E(^{4}He) *vs* the predictions for the ^{3}H binding energy for several interaction models. Results without (crosses) and with (diamonds) 3N forces are shown. The experimental point is marked by a star. The line represents a least square fit to NN force predictions only.



Elastic Nd scattering

Differential cross section in elastic pd and nd scattering at 65 and 135 MeV. The blue (violet) shaded bands are NN force only (NN+3NF) predictions for various interactions.





Ab Initio Calculations of Light Nuclei

Nuclear Interactions

…binding energies, excitation spectra, densities, transition amplitudes, low-energy astrophysical reactions, in terms of nucleons interacting with realistic potentials.

The nuclear Hamiltonian:

$$H = \sum_i T_i + \sum_{i<j} v_{ij} + \sum_{i<j<k} V_{ijk} + \cdots$$

Non-relativistic kinetic energy

NN potential

NNN potential

Argonne v18 (AV18) NN potential → can be written as a sum of eighteen operators (N = 18):

$$v_{ij} = \sum_{p=1}^{\mathcal{N}} v^p(r_{ij}) \mathcal{O}^p_{ij} \,,$$

$$v_{ij} = v^\gamma_{ij} + v^\pi_{ij} + v^I_{ij} + v^S_{ij} \,.$$

Electromagnetic

One-pion exc.

Intermediate range

Short range

One-pion exchange: $v_{ij}^{\pi} = \left[v_{\sigma\tau}^{\pi}(r)\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + v_{t\tau}^{\pi}(r)\,S_{ij}\right]\boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j + \left[v_{\sigma T}^{\pi}(r)\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j + v_{tT}^{\pi}(r)\,S_{ij}\right]T_{ij}$

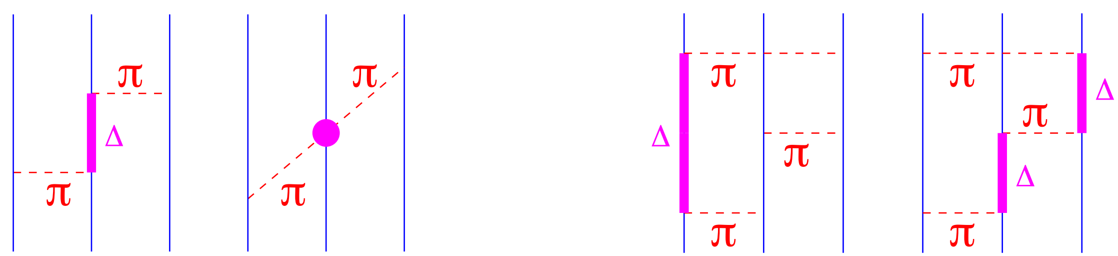
Operators: $\mathcal{O}_{ij}^{p=1-8} = \left[1, \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, S_{ij}, \mathbf{L}\cdot\mathbf{S}\right]\otimes\left[1, \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right]$

$\mathcal{O}_{ij}^{p=9-14} = \left[\mathbf{L}^2, \mathbf{L}^2\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, (\mathbf{L}\cdot\mathbf{S})^2\right]\otimes\left[1, \boldsymbol{\tau}_i\cdot\boldsymbol{\tau}_j\right]$ …quadratic in L.

$\mathcal{O}_{ij}^{p=15-18} = \left[T_{ij}, T_{ij}\,\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, T_{ij}\,S_{ij}, \tau_{i,z}+\tau_{j,z}\right]$ … break charge-independence.

The AV18 interaction has 42 parameters adjusted to NN data (χ^2/datum = 1.1).

The Illinois three-nucleon potential consists of two- and three-pion terms and a simple phenomenological repulsive term. Parameters adjusted to fit 17 nuclear levels for A ≤ 8.



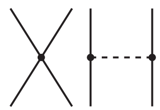
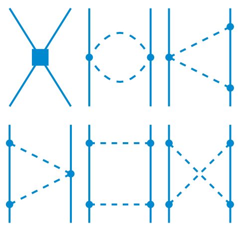
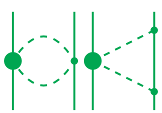
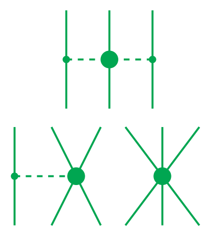
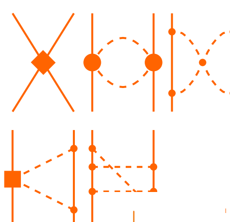
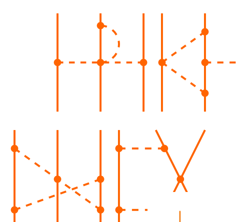
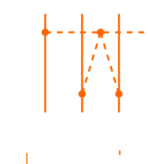
In light nuclei: $\langle V_{ijk}\rangle \sim (0.02 \text{ to } 0.09)\langle v_{ij}\rangle \sim (0.15 \text{ to } 0.6)\langle H\rangle$

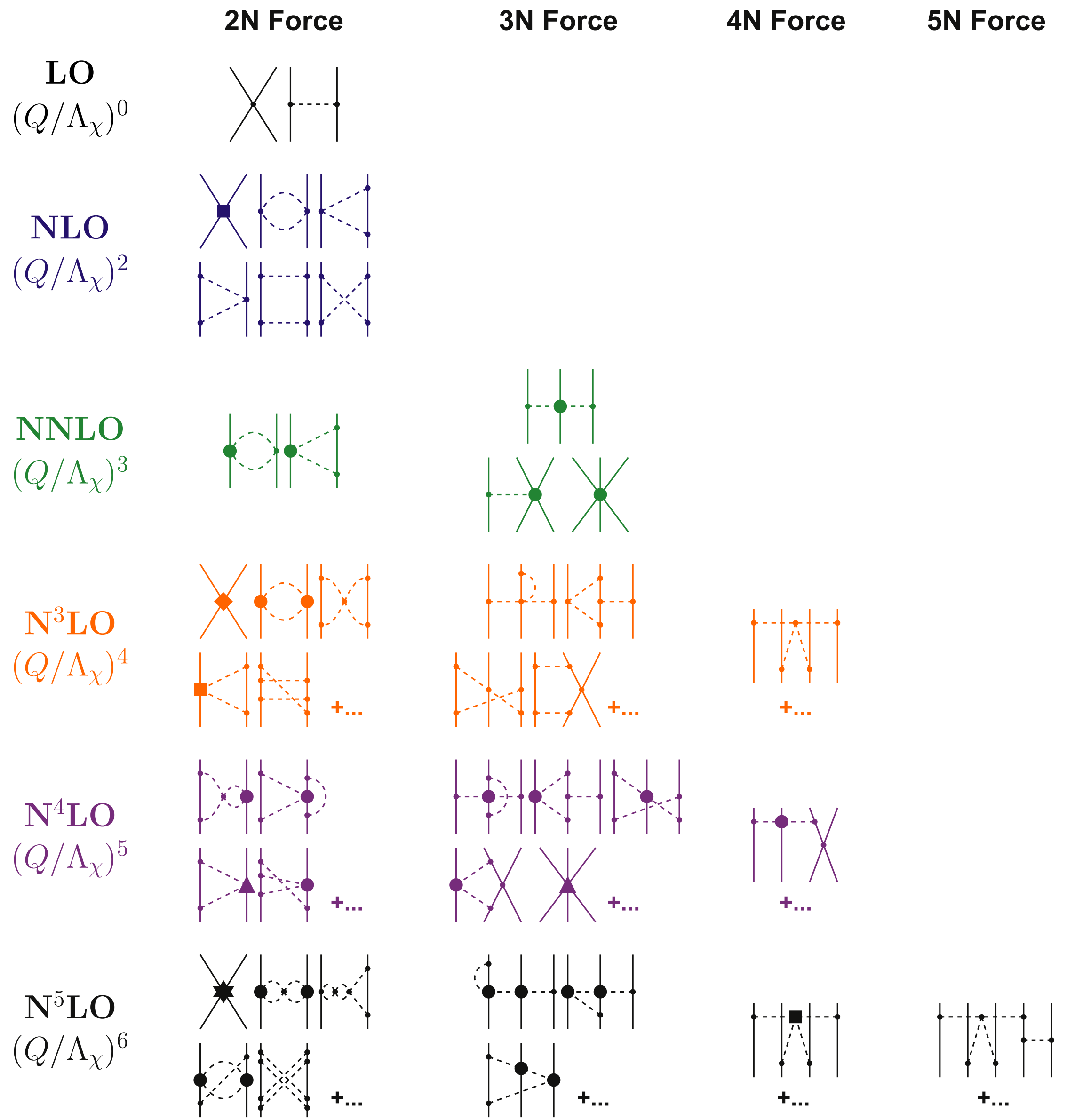
… because of a large cancellation of T and v_{ij}.

Chiral EFT forces

$$V = \sum_{\nu=0}^{\infty} V^{\nu}(\{C_i^{\nu}\}) \left(\frac{p}{\Lambda_b}\right)^{\nu}$$

… an infinite series of interaction terms. A power-counting scheme (ChPT) arranges the terms according to their importance, in powers of p/Λ_b. Typical momentum scale p (or pion mass), and chiral symmetry breaking scale Λ_b.

	NN	3N	4N
LO $(Q/\Lambda_\chi)^0$		$V_{\text{cont}}^{\nu=0} = C_{\mathbb{1}}\mathbb{1} + C_\sigma \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 + C_\tau \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 + C_{\sigma\tau} \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$ $V_\pi^{\nu=0}(\mathbf{p}, \mathbf{p}') = -\left(\frac{g_A}{2f_\pi}\right)^2 \frac{\boldsymbol{\sigma}_1 \cdot \mathbf{q}\, \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{\mathbf{q}^2 + m_\pi^2} \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2$	
NLO $(Q/\Lambda_\chi)^2$			
NNLO $(Q/\Lambda_\chi)^3$			
N^{3}LO $(Q/\Lambda_\chi)^4$			



2N Force
3N Force
4N Force
5N Force
LO
$(Q/\Lambda_\chi)^0$
NLO
$(Q/\Lambda_\chi)^2$
NNLO
$(Q/\Lambda_\chi)^3$
N^{3}LO
$(Q/\Lambda_\chi)^4$
N^{4}LO
$(Q/\Lambda_\chi)^5$
N^{5}LO
$(Q/\Lambda_\chi)^6$
+...

Quantum Monte Carlo Calculations

Variational Monte Carlo (VMC)

… start with a trial wave function which contains a number of variational parameters. These parameters are varied to minimize the expectation value of the Hamiltonian:

$$E_V = \frac{\langle \Psi_T | H | \Psi_T \rangle}{\langle \Psi_T | \Psi_T \rangle} = \frac{\sum_{\sigma\tau} \int \mathrm{d}\mathbf{R} \Psi_T^\dagger(\mathbf{R}, \sigma, \tau) H \Psi_T(\mathbf{R}, \sigma, \tau)}{\sum_{\sigma\tau} \int \mathrm{d}\mathbf{R} \Psi_T^\dagger(\mathbf{R}, \sigma, \tau) \Psi_T(\mathbf{R}, \sigma, \tau)} \geq E_0$$

ground-state energy!

$$\mathbf{R} = \{\mathbf{r}_1 \ldots \mathbf{r}_N\}, \sigma = \{\sigma_1 \ldots \sigma_N\}, \text{and } \tau = \{\tau_1 \ldots \tau_N\}$$

→ numerical evaluation of a multidimensional integral!

→ In the VMC a probability distribution *P* is used to sample a set of *M* configurations in {**R**, σ , τ } space that are used to estimate the integral:

$$E_V = \frac{\sum_{\sigma\tau} \int \mathrm{d}\mathbf{R} P(\mathbf{R}, \sigma, \tau) \frac{H \Psi_T(\mathbf{R}, \sigma, \tau)}{\Psi_T(\mathbf{R}, \sigma, \tau)}}{\sum_{\sigma\tau} \int \mathrm{d}\mathbf{R} P(\mathbf{R}, \sigma, \tau)}$$

$$|\Psi_T\rangle = \Big[\mathcal{S}\prod_{i<j}(1 + U_{ij} + \Sigma_k U_{ijk})\Big]\prod_{i<j} f_c(r_{ij})|\Phi\rangle$$

symmetrizer

2- and 3-body correlations

central (short-ranged repulsion) correlation

fully antisymmetric

$$U_{ij} = \sum_{p=2,6} u_p(r_{ij}) O^p_{ij}$$

The radial correlations $f_c(r)$and $u_p(r)$ include variational parameters that are chosen in order to minimize the energy E_V .

The one-body part of the trial wave function is a 1ħω shell-model wave function. It determines the quantum numbers of the state being computed and is fully antisymmetric. For ^{3}H and ^{3,4}He, can be antisymmetrized in just spin-isospin space, for example:

$$|\Phi(^3H, M_J = \tfrac{1}{2})\rangle = \frac{1}{\sqrt{6}}\big(|p\!\uparrow n\!\uparrow n\!\downarrow\rangle - |p\!\uparrow n\!\downarrow n\!\uparrow\rangle + |n\!\downarrow p\!\uparrow n\!\uparrow\rangle$$
$$- |n\!\uparrow p\!\uparrow n\!\downarrow\rangle + |n\!\uparrow n\!\downarrow p\!\uparrow\rangle - |n\!\downarrow n\!\uparrow p\!\uparrow\rangle\,\big)$$

Green's Function Monte Carlo

The VMC trial wave functions contain admixtures of excited-state components in addition to the desired exact ground-state component Ψ_0

$$\Psi_T = \Psi_0 + \sum \alpha_i \Psi_i$$

Green's Function Monte Carlo – projects Ψ_0 out of Ψ_T by propagating in imaginary time:

$$\begin{aligned} \Psi(\tau) &= \exp[-(H - \tilde{E}_0)\tau]\Psi_T \ , \\ &= e^{-(E_0 - \tilde{E}_0)\tau} \times [\Psi_0 + \sum \alpha_i e^{-(E_i - E_0)\tau} \Psi_i] \\ \lim_{\tau \to \infty} \Psi(\tau) &\propto \Psi_0 \ , \end{aligned}$$

$\tilde{E}_0$ - guess for the exact energy E_0

… evaluation of $\Psi(\tau)$ by introducing a small time step $\Delta\tau$, $\tau = n\Delta\tau$

$$\Psi(\tau) = \left[e^{-(H - E_0)\triangle\tau}\right]^n \Psi_T = G^n \Psi_T$$

G is the short-time Green's function.

$$G_{\alpha\beta}(\mathbf{R}', \mathbf{R}) = \langle \mathbf{R}', \alpha | e^{-(H - E_0)\triangle\tau} | \mathbf{R}, \beta \rangle$$

spin-isospin

$$\Psi(\mathbf{R}_n,\tau) = \int G(\mathbf{R}_n,\mathbf{R}_{n-1})\cdots G(\mathbf{R}_1,\mathbf{R}_0)\Psi_T(\mathbf{R}_0)\ d\mathbf{P}$$

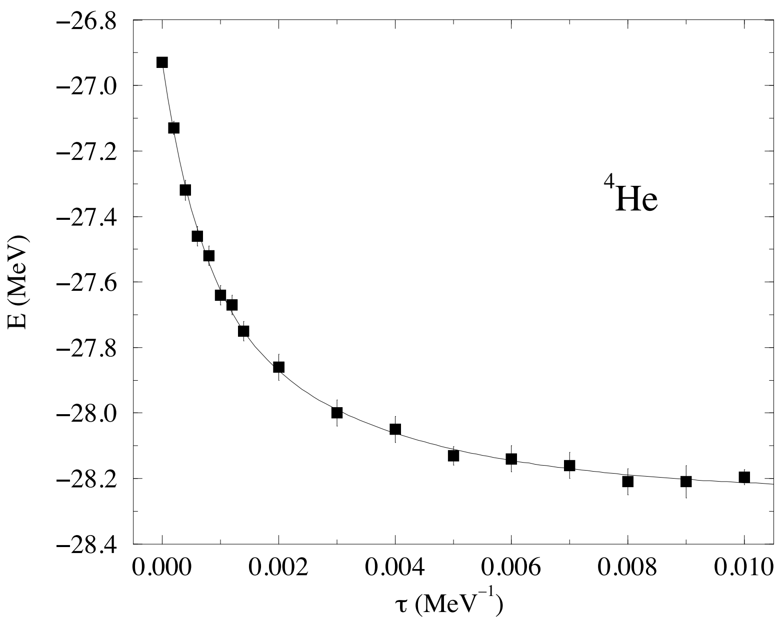
... includes spin and isospin degrees of freedom.

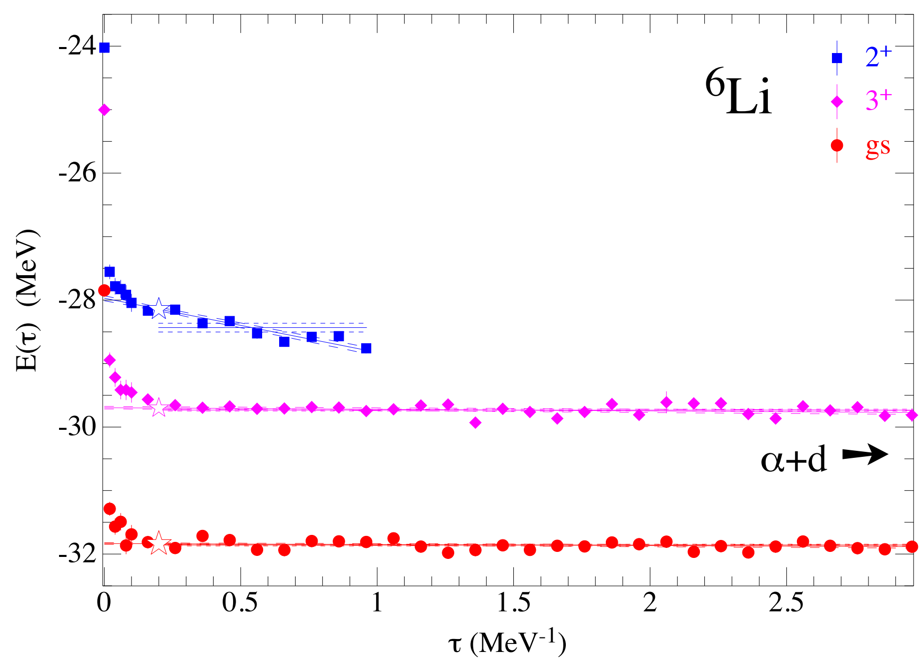
$$E(\tau) = \frac{\int \Psi_T^\dagger(\mathbf{R}_n)\ G^\dagger(\mathbf{R}_n,\mathbf{R}_{n-1})\cdots G^\dagger(\mathbf{R}_1,\mathbf{R}_0)\ H\ \Psi_T(\mathbf{R}_0)\ d\mathbf{P}}{\int \Psi_T^\dagger(\mathbf{R}_n)\ G^\dagger(\mathbf{R}_n,\mathbf{R}_{n-1})\cdots G^\dagger(\mathbf{R}_1,\mathbf{R}_0)\ \Psi_T(\mathbf{R}_0)\ d\mathbf{P}}$$

$$d\mathbf{P} = d\mathbf{R}_0 d\mathbf{R}_1\cdots d\mathbf{R}_n$$

Monte Carlo evaluation of the 3An integral.

Examples of GFMC propagation

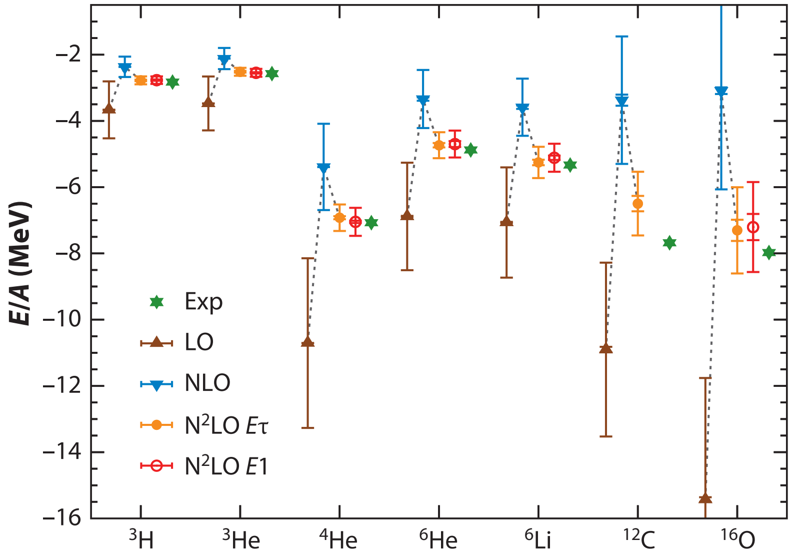


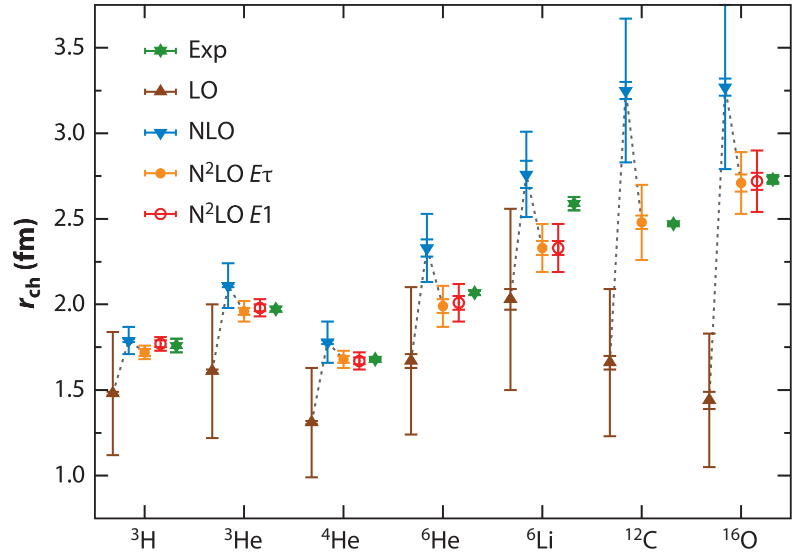


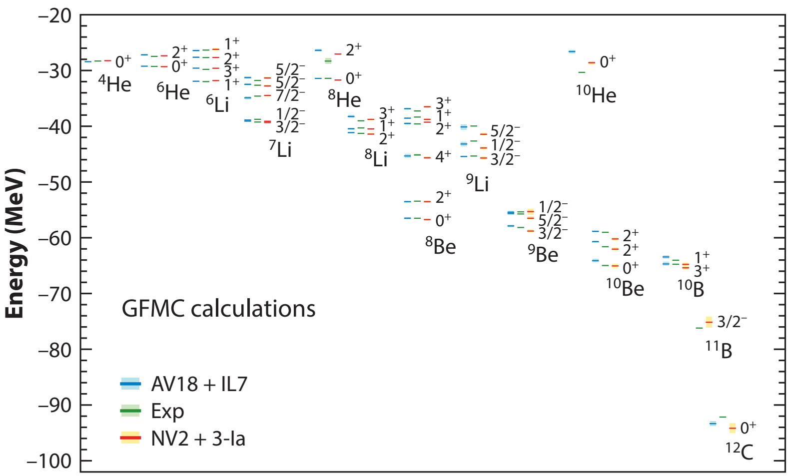
Illustrative results

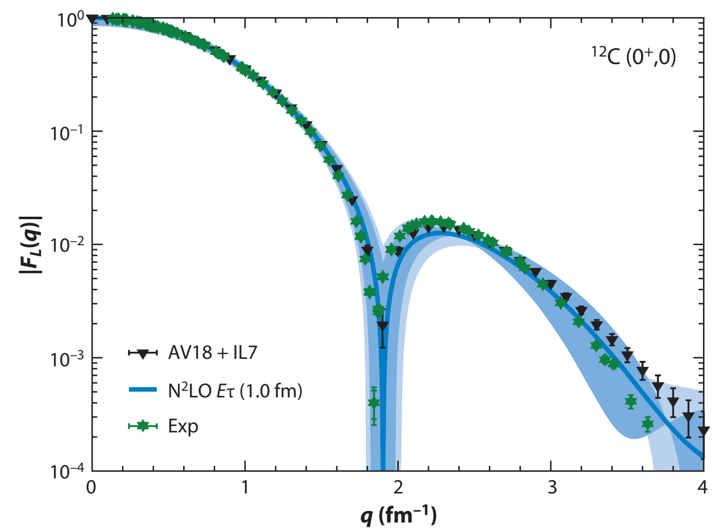
Annu. Rev. Nucl. Part. Sci. 2019. 69:279–305

Ground-state energies and charge radii for light nuclei with $3 \le A \le 16$ using $N\,N$ and $3N$ chiral EFT interactions.









Ground- and excited-state energies for light nuclei.

The longitudinal electric form factors for ^{12}C.

Ab initio No Core Shell Model

…system of A point-like nonrelativistic nucleons bound by realistic two- (NN) or two- plus three-nucleon (NNN) interactions. All the nucleons are considered active → 'no core'.

The Hamiltonian

$$H_A = T_{\rm rel} + \mathcal{V} = \frac{1}{A}\sum_{i<j}\frac{(\vec{p}_i - \vec{p}_j)^2}{2m} + \sum_{i<j}^{A} V_{{\rm NN},ij} + \sum_{i<j<k}^{A} V_{{\rm NNN},ijk},$$

… in NCSM, large but finite HO basis. An effective interaction appropriate for the basis truncation must be derived.

The Basis

A single-nucleon HO wave function can be written as: $\varphi_{nlm}(\vec{r}; b) = R_{nl}(r; b) Y_{lm}(\hat{r})$

HO length parameter: $b = \sqrt{\frac{\hbar}{m\Omega}}$

Slater determinant basis constructed from single-nucleon HO wave functions:

$$\varphi_{nljmm_t}(\vec{r}, \sigma, \tau; b) = R_{nl}(r; b)(Y_l(\hat{r})\chi(\sigma))_m^{(j)}\chi(\tau)_{m_t}$$

$$\sum_{i=1}^{A}(2n_i + l_i) \le N_{totmax}$$

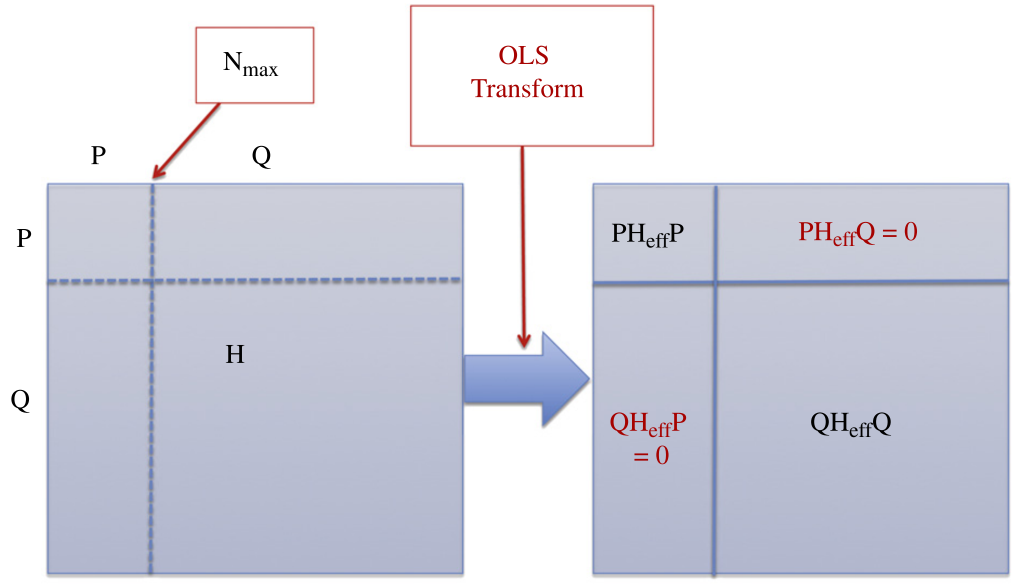
Effective interactions

In *ab initio* NCSM calculations a truncated HO basis is used. The inter-nucleon interactions act in the full space and, therefore, need to be renormalized in the truncated space, or model space. An effective Hamiltonian is constructed with the bare inter-nucleon interactions replaced by effective interactions.

… an arbitrary Hamiltonian: $H|k\rangle = E_k|k\rangle$

The full space is divided into the model space defined by a projector P and the complementary space defined by a projector Q , P + Q = 1.

The hermitian effective Hamiltonian can be obtained directly by a unitary transformation of the original Hamiltonian: $\bar{H}_{\text{eff}} = Pe^{-S}He^{S}P$



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The effective Hamiltonian contains many-body terms. For an A-nucleon system all terms up to A-body will in general appear in the effective Hamiltonian even if the original Hamiltonian consists of just two-body or two- plus three-body terms.

The two-body or three-body effective interaction is by construction exact for the two- or three-nucleon system. It is an approximation of the exact A-nucleon effective interaction.

The two-body effective Hamiltonian used in the A-nucleon system:

$$H_{A,\text{eff}}^{\Omega} = \sum_{i=1}^{A} h_i + \sum_{i<j}^{A} V_{2\text{eff},ij}.$$

The three-body effective Hamiltonian:

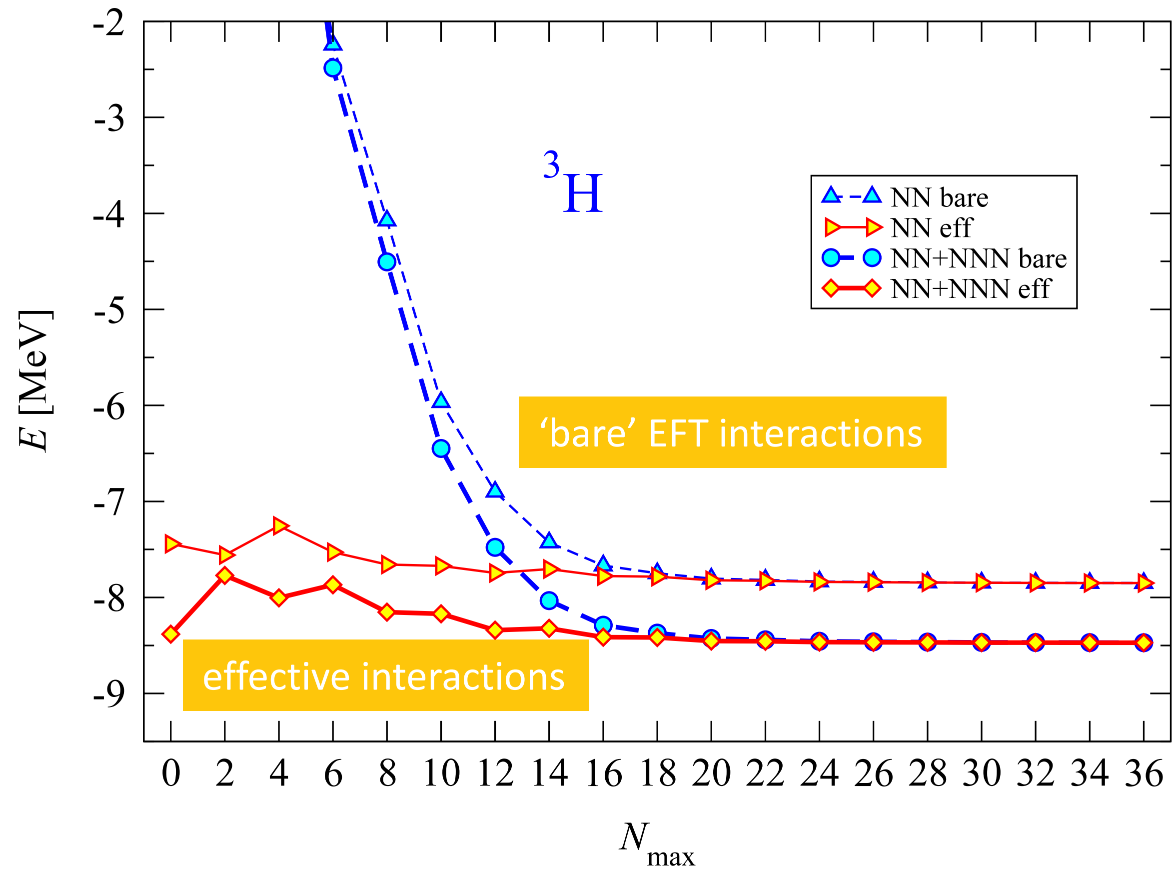
$$H_{A,\text{eff}}^{\Omega} = \underbrace{\sum_{i=1}^{A} h_i}_{\text{HO}} + \underbrace{\frac{1}{A-2} \sum_{i<j<k}^{A} V_{3\text{eff},ijk}^{\text{NN}}}_{\text{3-body effective interaction contribution from the NN interaction}} + \underbrace{\sum_{i<j<k}^{A} V_{3\text{eff},ijk}^{\text{NNN}}}_{\text{3-body effective interaction contribution from the NNN interaction}}$$

… plus H_{CM} must be subtracted.

The unitary transformation performed on the Hamiltonian should also be applied to other operators that are used to calculate observables.

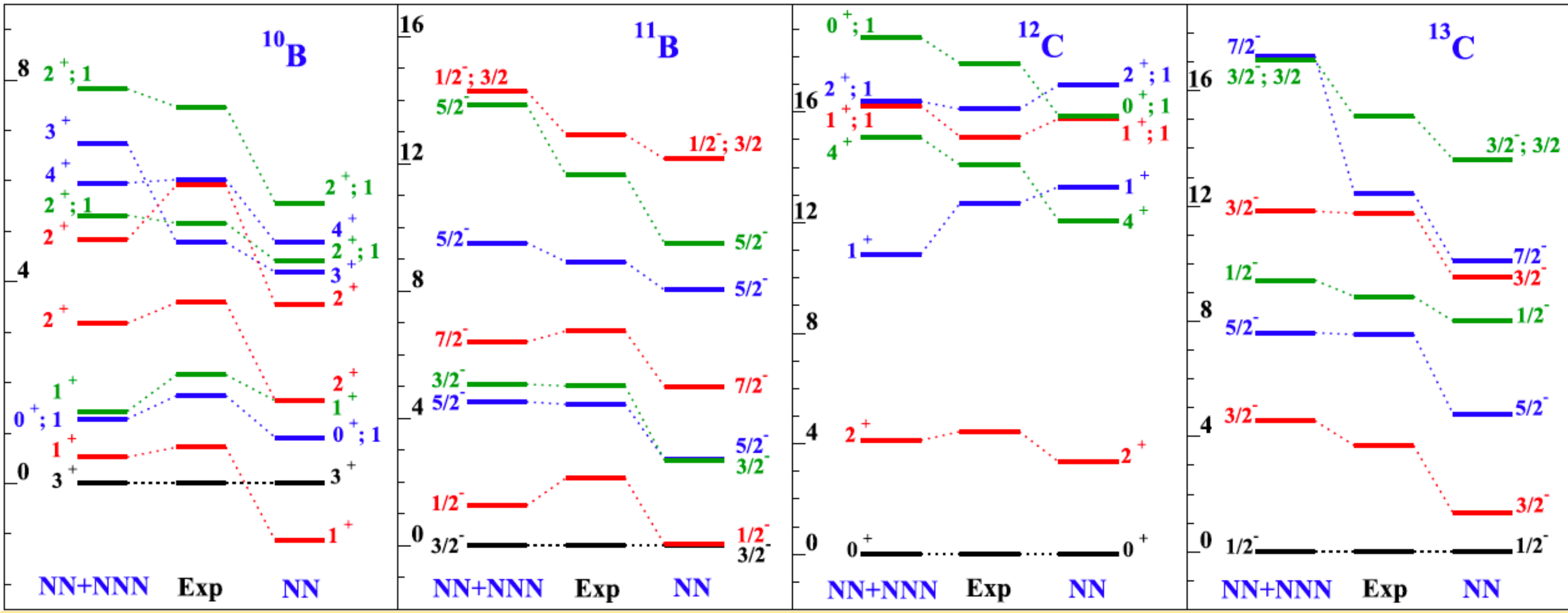
Convergence

^{3}H ground-state energy dependence on the size of the basis. The HO frequency: ħΩ=28 MeV.



The calculation without the NNN interaction converges to the ground-state energy -7.85 MeV. With the NNN interaction included, the result is -8.47 MeV, close to experiment (-8.48 MeV).

No Core Shell Model Applications



States dominated by 0p-shell configurations for ^{10}B, ^{11}B, ^{12}C, and ^{13}C calculated at N_{max} = 6, $\hbar\Omega$= 15 MeV. The excitation energy scales are in MeV.

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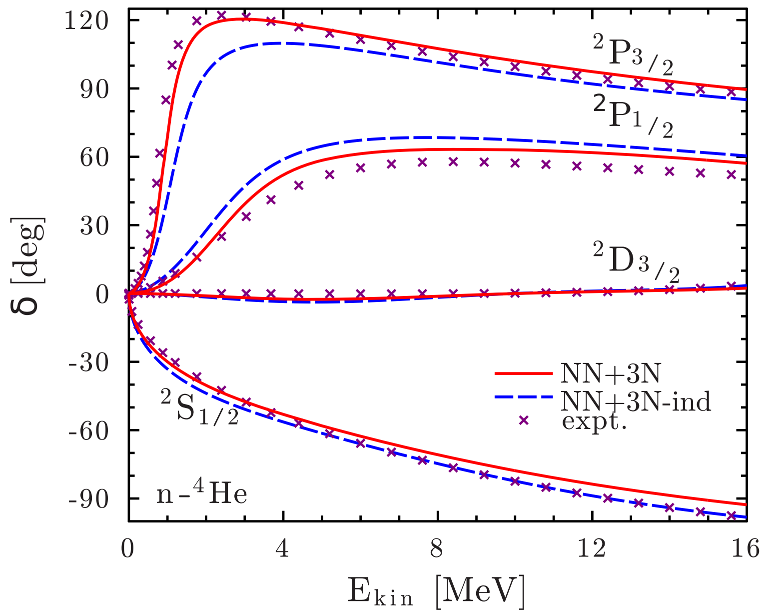
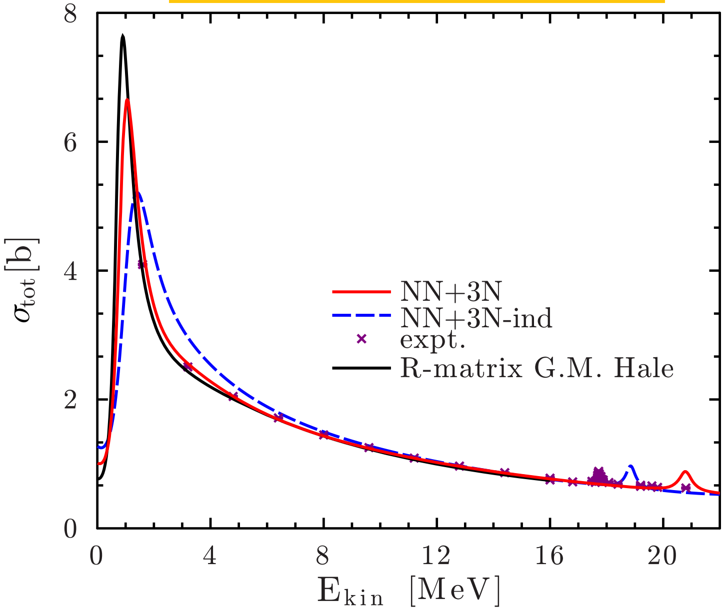
NCSM applications to nuclear reactions

Nuclei → bound states, unbound resonances, and scattering states.

^{4}He(n, n)^{4}He phase shifts



^{4}He(n, n)^{4}He cross section



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For more information please visit:
http://bela.phy.hr/quantixlie/hr/
https://strukturnifondovi.hr/

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I INVESTICIJSKI FONDOVI

Operativni program
KONKURENTNOST
I KOHEZIJA

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